E.A. OKAL *Geological Sciences 462*

1 May 2016

Chapter 4

Lamb's Problem: From Weyl's Integral to Cagniard–de Hoop's Method

Aki and Richards [1980]; *Chapter 6.*

The goal of this chapter is to investigate the interaction of a spherical wavefront with a planar boundary. Obviously, in the limit of short wavelengths, and in the near field, it describes the case of a point source close to the Earth's surface.

1. Weyl's Integral

This problem considers a monochromatic spherical wave whose scalar potential ϕ can be written

$$\phi(x, y, z; t) = \frac{1}{R} \cdot \exp\left(i \frac{\omega R}{c} \right) \cdot e^{-i\omega t} \tag{1}$$

Note that we will consider here both a spherical and a cylindrical system of coordinates, with

$$R = \sqrt{x^2 + y^2 + z^2} \; ; \qquad r = \sqrt{x^2 + y^2} \tag{2}$$

Following the results in Chapter 2, we will consider a Green's function, such that [*A&R* (6.2)]

$$\Delta \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 4\pi \, \delta(\mathbf{x}) \, e^{-i\omega t} \tag{3}$$

Along the lines of Chapter 3, we consider the 3-dimensional Fourier transform

$$\phi(x, y, z; t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \bar{\phi}(k_x, k_y, k_z) \, e^{i\mathbf{k}\cdot\mathbf{x}} \cdot e^{-i\omega t} \cdot d^3\mathbf{k} \tag{4}$$

then $\bar{\phi}$ has to satisfy

$$\bar{\phi}(k_x, k_y, k_z) = 1 \cdot \frac{4\pi}{\mathbf{k}^2 c^2 - \omega^2} \tag{5}$$

since 1 is the Fourier transform of $\delta(\mathbf{x})$. We thus come to the conclusion that

$$\frac{1}{R} \exp\left(i \frac{\omega R}{c} \right) = \frac{1}{2\pi^2} \int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_y \int_{-\infty}^{+\infty} \frac{\exp(i\,\mathbf{k}\cdot\mathbf{x})}{\mathbf{k}^2 - \omega^2 / c^2} \, dk_z \tag{6}$$

This is a perfectly legitimate expression, but on the other hand, we know that physically, only certain combinations of $\{k_x, k_y, k_z\}$ should contribute to the integral, since the vector $\mathbf{k}$ must satisfy the wave equation:

$$\mathbf{k}^2 = k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2} \tag{7}$$

For this purpose, we focus on the integral over k_z :

$$I(k_x, k_y; z) = \int_{-\infty}^{+\infty} \frac{e^{i k_z z}}{k_x^2 + k_y^2 + k_z^2 - \omega^2 / c^2} \cdot dk_z \tag{8}$$

In order to compute this integral, we first deform it in the complex plane (assuming that k_z is a fully complex variable), and in addition, we introduce a small negative imaginary value to c, so that it becomes

$$c = c^{real}(1 - i\varepsilon) \qquad 0 < \varepsilon \ll 1 \tag{9}$$

$$\frac{1}{c} = \frac{1}{c^{real}} \cdot (1 + i\varepsilon)$$

[The meaning of this transformation is to take into account a small amount of *attenuation*. In the end, the results can be taken in the limit $\varepsilon \to 0$.]

- *Assuming first that* $z > 0$,

  we then compute the integral I along the contour Γ shown on Figure 1. Along the half-circle, we set $k_z = k_0 e^{i\psi}$, ψ varying from 0 to π, and let $k_0 \to \infty$. For sufficiently large k_0, the modulus of the denominator grows like k_0^2, whereas the numerator is at most of order k_0 (this is where $z > 0$ is crucial so that the exponential does not blow up). Hence the contribution of the arc to the integral I goes to 0 as $k_0 \to \infty$.

  The integrand in I has one pole k_z^P inside the contour, where the denominator can be written

$$k_z^2 - (\omega^2 / c^2 - k_x^2 - k_y^2) = (k_z - k_z^P) \cdot (k_z + k_z^P) \tag{10}$$

  and by application of the residue theorem, the integral I then becomes:

$$I(k_x, k_y; z) = 2 i \pi \cdot \frac{\exp(i k_z^P z)}{2 k_z^P} \tag{11}$$

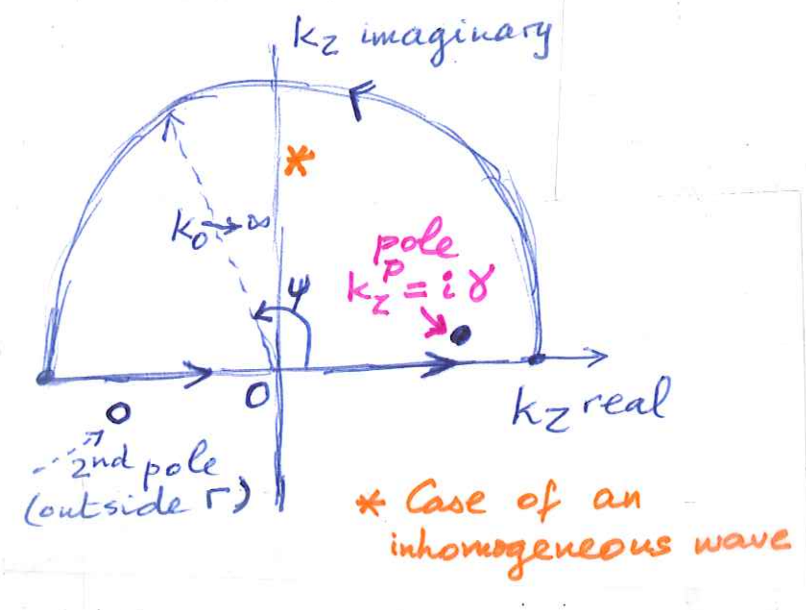
  We now change the notation (slightly !) by setting

$$k_z^P = i\gamma \tag{12a}$$

  Because we have taken $\varepsilon > 0$ in (9), it follows that

$$\mathrm{Re}(\gamma) > 0 \tag{12b}$$

Figure 1.



There follows from (11) that

$$I\,(k_x\,,\,k_y\,;\,z) \;=\; \pi\,\frac{\exp(-\gamma\,z)}{\gamma} \tag{13}$$

The meaning of (13) is that there is, in (8), and hence in (6), a contribution for only one value of k_z, namely that which satisfies the wave equation (7).

This brings in the very powerful concept of

DUALITY BETWEEN POLES OF INTEGRATION AND INDIVIDUAL SEISMIC PHASES

→ Note on Figure 1

- First, that it has been drawn in the case of a "homogeneous" wave, where the pole is (nearly; except for ε) on the real axis, corresponding to

$$k_z^P \;=\; \sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2} \tag{14}$$

In the inhomogeneous case where ($k_x^2 + k_y^2 > \omega^2 / c^2$), then the pole is (nearly) on the imaginary axis (shown as the asterisk on Figure 1), and given by

$$k_z^P \;=\; i\,\sqrt{k_x^2 + k_y^2 - \frac{\omega^2}{c^2}} \tag{15}$$

- There is a second pole (shown as the open circle on the Figure), but with $\varepsilon > 0$, it lies outside the contour Γ and does not contribute to I.

- If we now consider *negative* values of z, the exponential along the half-circle will blow up and the contribution of the arc is no longer negligible. We then take the contour Γ', symmetric of Γ with respect to the real axis, which will have only one pole, namely $-k_z^P$ (open circle on Figure 1). If we take the same definition of γ, namely

$$\gamma^2 \;=\; k_x^2 + k_y^2 - \frac{\omega^2}{c^2}\;; \qquad \mathrm{Re}\,(\gamma) > 0\;, \tag{16}$$

the pole is now at $(-i\gamma)$ and the residue theorem leads to

$$I\,(k_x\,,\,k_y\,;\,z) \;=\; -2\,i\,\pi\,\frac{\exp(\gamma\,z)}{-2\,i\,\gamma} \;=\; \pi\,\frac{\exp(\gamma\,z)}{\gamma} \tag{17}$$

(the extra minus sign comes from the fact that the contour is now traveled clockwise), so that in all cases, (13; $z > 0$) or (17; $z < 0$), I can be written

$$I\,(k_x\,,\,k_y\,;\,z) \;=\; \pi\,\frac{\exp(-\gamma\,|z|)}{\gamma} \tag{18}$$

which leads to

$$\frac{1}{R}\,e^{i\frac{\omega R}{c}} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_y \cdot \frac{e^{i(k_x x + k_y y) - \gamma|z|}}{\gamma} \qquad (19)$$

with, again, γ given by (16).

> *THIS CONSTITUTES WEYL's DECOMPOSITION OF*
> *A SPHERICAL WAVE INTO PLANE WAVES*
>
> *Note that it involves both*
> *HOMOGENEOUS ("Body") and INHOMOGENEOUS ("Surface") waves*

2. Sommerfeld's Decomposition

Sommerfeld's decomposition consists of transforming (19) into a single integration of *cylindrical* waves. We set

$$x = r\cos\phi\;; \qquad\qquad y = r\sin\phi \qquad (20)$$

and

$$k_x = k_r\cos\phi' \qquad\qquad k_y = k_r\sin\phi' \qquad (21)$$

(Note that the angle ϕ has nothing to do with the potential ϕ in (1)), and substitute into (19)

$$\frac{1}{R}\,e^{i\frac{\omega R}{c}} = \frac{1}{2\pi}\int_0^{+\infty}\frac{k_r}{\gamma}\,dk_r\int_0^{2\pi} e^{i(k_r r\cos(\phi-\phi')) - \gamma|z|}\,d\phi' \qquad (22)$$

(Note that γ is independent of ϕ' and thus can be taken out of the last integral.)

We recall [*Abramowitz and Stegun,* (9.1.18) p. 360]

$$\int_0^{2\pi}\exp(i\,x\cos\phi)\,d\phi = 2\pi\,J_0(x) \qquad (23)$$

where J_0 is the Bessel function of order 0. Hence

$$\frac{1}{R}\,e^{i\frac{\omega R}{c}} = \int_0^{+\infty}\frac{k_r\,J_0(k_r r)\exp(-\gamma|z|)}{\gamma}\cdot dk_r \qquad (24)$$

with, again, (16) expressed in cylindrical polars,

$$\gamma^2 = k_r^2 - \frac{\omega^2}{c^2}\;; \qquad\qquad \mathrm{Re}(\gamma) > 0\,, \qquad (16r)$$

We have decomposed the spherical wave onto "cylindrical waves" (see end of Chapter 2), attenuated along z, and weighted by k_r/γ.

This result (24) is known as Sommerfeld's Integral

3. Interaction with a discontinuity.

We now attack the problem of the interaction of a spherical wave such as (1) with a planar discontinuity, by making use of a Sommerfeld decomposition (16).

Before we can do so, we shall prove a

LEMMA

For a surface of discontinuity perpendicular to Oz, the reflection / transmission coefficients of **cylindrical waves** *of wavenumber k_r are the same as those of plane waves of horizontal slowness*

$$p = \frac{k_r}{\omega} \tag{23}$$

PROOF

We consider a cylindrical wave whose potential has the form (not writing the time factor $e^{-i\omega t}$):

$$\frac{1}{R} e^{i\frac{\omega R}{c}} = \int_0^{+\infty} \frac{k_r\, J_0(k_r r) \exp(-\gamma |z|)}{\gamma} \cdot dk_r \tag{24}$$

By reading (23) backwards, we can expand it as

$$\frac{1}{R} e^{i\frac{\omega R}{c}} = \frac{1}{2\pi} \int_0^{+\infty} \frac{k_r}{\gamma}\, dk_r \int_0^{2\pi} e^{i(k_r r \cos(\phi-\phi') - \gamma|z|}\, d\phi' \tag{22}$$

or even, if we consider a cartesian coordinate system, as

$$\frac{1}{R} e^{i\frac{\omega R}{c}} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_y \cdot \frac{e^{i(k_x x + k_y y) - \gamma|z|}}{\gamma} \tag{19}$$

For each of combination of k_x and k_y in (19), we have a plane wave of amplitude

$$\frac{1}{2\pi\gamma} e^{i(k_x x + k_y y) - \gamma|z|} \tag{25}$$

which interacts with a planar boundary perpendicular to $\hat{\mathbf{e}}_\mathbf{z}$. We can always rotate the $\{x, y\}$ frame into a new frame $\{X, Y\}$ in which the vector $\mathbf{k}$ will have no k_Y component. Then its component k_X becomes

$$k_X = \sqrt{k_x^2 + k_y^2} = k_r \tag{26}$$

and the plane wave can be written as

$$\frac{1}{2\pi\gamma} e^{i k_X X - \gamma|z|} \tag{27}$$

That plane wave, with a horizontal wavenumber k_X, and hence a slowness $p = k_X / \omega = k_r / \omega$, interacts with a boundary perpendicular to the direction $\hat{\mathbf{e}}_\mathbf{z}$. According to Chapter 3, it generates reflected and transmitted plane waves, with appropriate amplitudes. For example, in the case of an incident P and a reflected P, we generate a plane wave of amplitude

$$\acute{P}\grave{P} \cdot \frac{1}{2\pi\gamma} e^{i k_X X + \gamma|z|} = \acute{P}\grave{P} \cdot \frac{1}{2\pi\gamma} e^{i(k_x + k_y y) + \gamma|z|} \qquad (28)\dagger$$

To obtain the full potential of the reflected wave, we must now integrate back over k_x and k_y

$$\int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_y \quad \acute{P}\grave{P} \cdot \frac{1}{2\pi\gamma} e^{i(k_x + k_y y) + \gamma|z|} \qquad (29)$$

We then implement the change of variables

$$x = r\cos\phi\ ; \qquad y = r\sin\phi \qquad (20)$$

and

$$k_x = k_r\cos\phi' \qquad k_y = k_r\sin\phi' \qquad (21)$$

to obtain

$$\int_0^{+\infty} dk_r \int_0^{2\pi} d\phi' \quad \acute{P}\grave{P} \cdot \frac{k_r}{2\pi\gamma} e^{i k_r r\cos(\phi-\phi') + \gamma|z|} \qquad (30)$$

and because $\acute{P}\grave{P}$ *depends only on* p*, and hence on* k_r*, it is independent of* ϕ'*, and so the whole term* $\acute{P}\grave{P} \cdot \frac{k_r}{2\pi\gamma}$ *can be taken out of the* ϕ' *integral, leading to*

$$\int_0^{+\infty} dk_r \quad \acute{P}\grave{P} \cdot \frac{k_r}{2\pi\gamma} e^{\gamma|z|} \cdot \int_0^{2\pi} e^{i k_r r\cos(\phi-\phi')} \cdot d\phi' \qquad (31)$$

or, remembering (23)

$$\int_0^{+\infty} \acute{P}\grave{P} \cdot \frac{k_r J_0(k_r r)}{\gamma} e^{\gamma|z|} \cdot dk_r \qquad (32)$$

which is the integral of the individual terms in (24), with just the opposite dependence on z, and weighted by the coefficient $\acute{P}\grave{P}$, computed for the slowness (23).

Q. E. D.

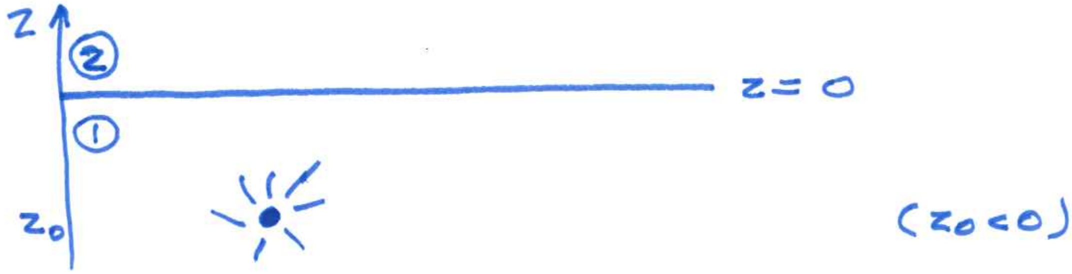
EVERYTHING TAKES PLACE AS IF THE "CYLINDRICAL WAVE" (24) *HAD REFLECTED WITH THE PLANAR COEFFICIENT* $\acute{P}\grave{P}$

The derivation of this important lemma is often overlooked and the result taken for granted.

† In these formulæ, $\acute{P}\grave{P}$ must be a reflection coefficient for *POTENTIALS,* but this does not change anything to the argument

- **Reflection at a plane boundary**

We now change the notation (again), and shift the boundary to $z = 0$, while we move the source to $z = z_0 < 0$.



We consider an incident wave in medium 1 with potential

$$\phi(R,t) = \frac{1}{R} \cdot \exp\left(\frac{R}{\alpha_1} - t\right) = \int_0^{+\infty} \frac{k_r}{\gamma} J_0(k_r r) \cdot e^{-\gamma|z-z_0|} \cdot e^{-i\omega t} \cdot dk_r \tag{33}$$

with

$$\gamma = -i\sqrt{\frac{\omega^2}{\alpha_1^2} - k_r^2} \quad ; \qquad \mathrm{Re}(\gamma) > 0 \tag{34}$$

We introduce the slowness p through

$$k_r = \omega p \tag{23}$$

yielding

$$\phi(R,t) = \int_0^{+\infty} \frac{J_0(\omega p r) \cdot e^{+i\omega|z-z_0|\sqrt{1/\alpha_1^2 - p^2}}}{-i\omega\sqrt{1/\alpha_1^2 - p^2}} \cdot e^{-i\omega t} \cdot \omega^2 p\, dp \tag{35a}$$

or

$$\phi(R,t) = \int_0^{+\infty} i\omega \cdot \frac{p J_0(\omega p r)}{\xi_1} \cdot e^{+i\omega\xi_1|z-z_0|} \cdot e^{-i\omega t} \cdot dp \tag{35b}$$

where we have introduced yet a new variable,

$$\xi_1 = \sqrt{\frac{1}{\alpha_1^2} - p^2} \quad ; \qquad \mathrm{Re}(\xi_1) > 0 \tag{36}$$

For $z = 0$ at the surface of discntinuity, this constitutes a decomposition in cylindrical waves, weighted by the coefficient

$$i\frac{p\omega}{\xi_1} \cdot e^{-i\omega\xi_1 z_0} \tag{37}$$

Applying the *LEMMA*, we infer that any wave resulting from the interaction with the boundary can be written as a superposition of cylindrical waves, weighted by the appropriate refelction or transmission coefficients. For example, the potential of the $P \rightarrow P$ reflected wave will be written as

$$\phi = i\,\omega \int_0^\infty \acute{P}\grave{P} \cdot \frac{p}{\xi_1} \cdot J_0(\omega p r) \cdot e^{-i\,\omega\,\xi_1 (z + z_0)} \cdot e^{-i\omega t} \cdot dp \qquad (38)\ (\text{A\&R } 6.12)$$

that of a transmitted $P \rightarrow P$ as

$$\phi = i\,\omega \int_0^\infty \acute{P}\acute{P} \cdot \frac{p}{\xi_1} \cdot J_0(\omega p r) \cdot e^{-i\,\omega (\xi_1 z_0 - \xi_2 z)} \cdot e^{-i\omega t} \cdot dp \qquad (39)\ (\text{A\&R } 6.13)$$

where

$$\xi_2 = \sqrt{\frac{1}{\alpha_2^2} - p^2} \quad ; \qquad \mathrm{Re}(\xi_2) > 0 \qquad (40)$$

In the case of *CONVERTED* waves, *e.g.,* a $P \rightarrow S$ transmitted wave, we will have a potential

$$\psi = i\,\omega \int_0^\infty \acute{P}\acute{S} \cdot \frac{p}{\xi_1} \cdot J_0(\omega p r) \cdot e^{-i\,\omega (\xi_1 z_0 - \eta_2 z)} \cdot e^{-i\omega t} \cdot dp \qquad (41)$$

where

$$\eta_2 = \sqrt{\frac{1}{\beta_2^2} - p^2} \quad ; \qquad \mathrm{Re}(\eta_2) > 0 \qquad (42)$$

while, for an $S \rightarrow P$ reflected wave, we would have

$$\phi = i\,\omega \int_0^\infty \acute{S}\grave{P} \cdot \frac{p}{\eta_1} \cdot J_0(\omega p r) \cdot e^{-i\,\omega (\eta_1 z_0 + \xi_1 z)} \cdot e^{-i\omega t} \cdot dp \qquad (43)$$

where

$$\eta_1 = \sqrt{\frac{1}{\beta_1^2} - p^2} \quad ; \qquad \mathrm{Re}(\eta_1) > 0 \qquad (44)$$

Remarks about Formulæ (38)–(43)

→ The choice of parameters ξ_1, ξ_2, η_1, η_2 in the exponentials is straightforward: the coefficient multiplying z_0 represents the propagation from the source, so it characterizes the *incident* wave, while the multiplier of z represents the emerging wave, hence the various combinations,

but

the coefficient in the denominator (in *red* in (43)) comes from the γ term in the expansion (6) of the incident wave before the reflection coefficients are applied, which itself comes all the way from the residue theorem... Hence, *THAT TERM IS ALWAYS CHARACTERISTIC OF THE INCIDENT WAVE.*

→ Because the various coeficients ($\acute{P}\grave{P}$ *et al.*) depend heavily on p, the integration over p in the integrals (38)–(43) is not trivial, and cannot be given an analytical form. This is why we cannot expect that it would lead to the reconstruction of a nice spherical wave of the form (1), even though an exponential of the form $e^{-i\omega\xi_1(z+z_0)}$ looks very much like something radiating away from an image located at $z=-z_0$. Yes, there is a kind of reflected image of the source, but the wave coming from it is *NOT* spherical, since its amplitude will be distorted as a function of p. In the case of a change of velocity (transmission into the other medium, or conversion to S upon reflection), there is not even a precise geometrical image, as is well known in optics.

The bottom line is, even though cylindrical waves appear to reflect/transmit while remaining cyclindrical, this property does not extend to spherical waves.

- **Trying to perform $\int dp$ for the reflected wave.**

We consider here the further simplified case of the boundary between two liquids (so we are no longer bothered by the velocities β, and by the possibility of conversions). We thus focus on a $P \rightarrow P$ reflection, whose potential will be given by (38):

$$\phi^{P,\,refl.} = i\,\omega\, e^{-i\omega t} \cdot \int_0^\infty \acute{P}\grave{P} \cdot \frac{p}{\xi_1} \cdot J_0(\omega p r) \cdot e^{-i\omega\xi_1(z+z_0)} \cdot dp \qquad (38)$$

The fundamental problems are that (i) ξ_1 will carry (at least for certain values of p) a real part, so that the argument of the exponential will oscillate (and faster as $\omega \rightarrow 0$); (ii) ξ_1 involves a complex square root, so that branch cuts will be present. As always in complex calculus, one attempts to rely on the deformation of the integral contour, but then (iii), we note that the integral extends only from 0 to $+\infty$, not from $-\infty$ to $+\infty$.

(iii) We first address the last point (iii), by noting that, at least, $\acute{P}\grave{P}$ and ξ_1 are *even* in p. We substitute the Bessel function for the Hankel ones:

$$J_0(x) = \frac{1}{2}\left[H_0^{(1)}(x) + H_0^{(2)}(x)\right] \quad \text{with} \quad H_0^{(2)}(x) = -H_0^{(1)}(-x) \qquad (45)$$

[*Abramowitz and Stegun,* (9.1.39), p.361], obtaining

$$\phi^{P,\,refl.} = \frac{i\,\omega}{2}\, e^{-i\omega t} \cdot \int_{-\infty}^{+\infty} \acute{P}\grave{P} \cdot \frac{p}{\xi_1} \cdot H_0^{(1)}(\omega p r) \cdot e^{-i\omega\xi_1(z+z_0)} \cdot dp \qquad (46)\ (A\&R\ 6.15)$$

(ii) *Branch cuts*

ξ_1 has two branch points at $p = \pm\dfrac{1}{\alpha_1}$, and we have imposed $\mathrm{Im}(\xi_1) > 0$, so we take the branch cut along the line $\mathrm{Im}(\xi_1) = 0$. Similarly for ξ_2 (note that even though ξ_2 does not appear directly in the integrals (38) or (46), it is present in $\acute{P}\grave{P}$ and thus its branch cuts will affect them). Finally, the Hankel function of first kind $H_0^{(1)}(x)$ has a branch point at $x=0$ (see its behavior on *Abramowitz and Stegun*'s Figure (9.4)), so that we put a branch cut along p real negative. Hence the diagram of the integration contour on Page 204 of *Aki & Richards* [1980], reproduced below.

204 REFLECTION AND REFRACTION OF SPHERICAL WAVES; LAMB'S PROBLEM

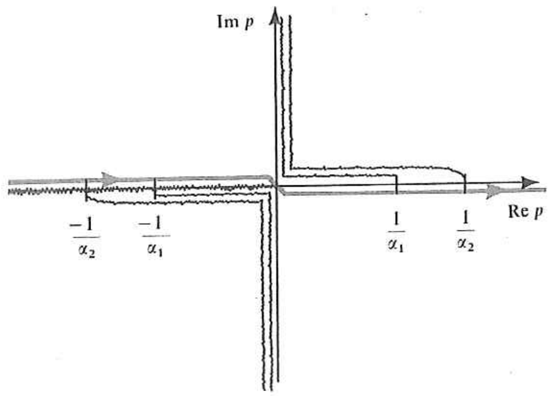


FIGURE **6.4**
Branch cuts for ξ_1, ξ_2, and $p^{1/2}$ in the complex p-plane. The cuts are given by $\text{Im}\,\xi_1 = 0$, $\text{Im}\,\xi_2 = 0$; and $\text{Re}\,p^{1/2} = 0$ (this being the cut assumed in (6.16). In fact, it is directly a branch cut for $H_0^{(1)}(\omega pr)$). The integration path for p^{refl} (see (6.17)) lies on the negative real axis just above three cuts, and lies on the positive real axis just below two cuts.

(i) The final challenge is to compute (46) taking into account the fast variation (with p, and especially at high frequency ω) of the complex argument of the exponential under the integral. We recognize in (46) the problem addressed in Chapter 1 in our study of the saddle-point approximation in the context of phase-stationary asymptotics.

We recall that we are dealing here with a high-frequency approximation ($\omega \to \infty$), so that we can replace the Hankel function by its asymptotic expansion [*Abramowitz and Stegun* (9.2) p.364]:

$$H_0^{(1)}(x) = \sqrt{\frac{2}{\pi x}} \cdot e^{i(x-\pi/4)} \cdot \left[1 - \frac{i}{8x} + O\left(\frac{1}{x^2}\right)\right] \tag{47}$$

which leads to

$$\phi^{P,\,refl.} = \sqrt{\frac{\omega}{2\pi r}} \cdot e^{-i(\omega t - \pi/4)} \cdot \int_{-\infty}^{+\infty} \acute{P}\grave{P}\, \frac{\sqrt{p}}{\xi_1} \cdot e^{i\omega(pr - \xi_1 z - \xi_1 z_0)} \cdot dp \tag{48}$$

This is exactly what we have described in Chapter 1, with

$$\begin{aligned} z &\quad \text{substituted with} \quad \omega \\ t &\quad \text{substituted with} \quad p \\ f(t) &\quad \text{substituted with} \quad i(pr - \xi_1 z - \xi_1 z_0) \end{aligned} \tag{49}$$

> *In the high-frequency limit $\omega \to \infty$,*
>
> *WE ARE GOING TO REPLACE THE INTEGRAL $\int dp$ WITH A SUM OF CONTRIBUTIONS AT THE SADDLE POINTS OF THE FUNCTION $f(t)$.*
>
> *EACH SADDLE POINT WILL CORRESPOND TO A SEISMIC "PHASE".*

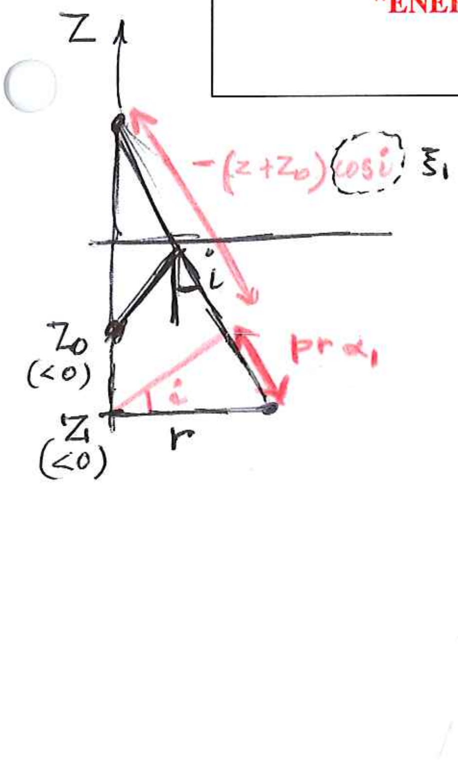
In the present case, there will be only one saddle point.

We note that the function f has the dimensions of a time, and, noting further that $z < 0$; $z_0 < 0$; $\xi_1 = \cos i / \alpha_1$, we can verify on Figure B that

$$pr - \xi_1 z - \xi_1 z_0 = T_{travel}. \tag{50}$$

the travel time along the reflected ray.

> *THE DISCRETIZATION OF THE INTEGRAL IS JUST A MATHEMATICAL EXPRESSION OF FERMAT's PRINCIPLE*
>
> **"ENERGY IS CARRIED ONLY ALONG THOSE RAYS WHICH EXTREMIZE THE TRAVEL TIME".**



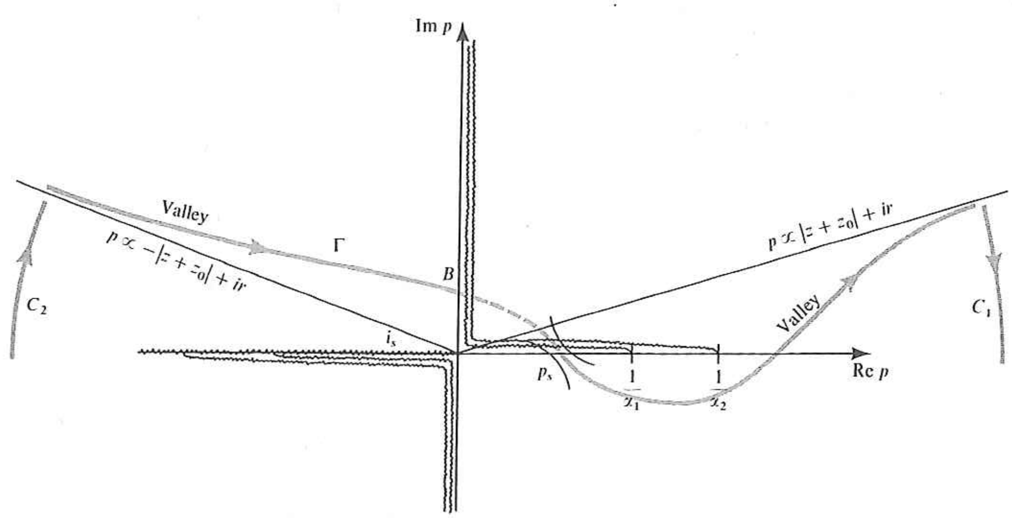


FIGURE **6.6**

The steepest-descents path Γ in the complex ray-parameter plane for obtaining P^{refl} (see (6.17)), when $\alpha_1 > \alpha_2$. We have indicated the exact path for its whole length, using the rule that $\omega f(p) - \omega f(p_s)$ is negative real, so that $\exp[\omega f(p)]$ decays exponentially away from the saddle point. Near the saddle point itself, ridges and valleys are as shown in Box 6.3, with $\chi = -\pi/4$. The dotted path is on a lower Riemann sheet, for which Im ξ_1 and Im ξ_2 are negative.

In the case of the reflected P wave, we have

$$f(p) = i\,(pr - \xi_1 z - \xi_1 z_0)\ ; \qquad \xi_1 = \sqrt{\frac{1}{\alpha_1^2} - p^2}\ ; \qquad \frac{d\,\xi_1}{dp} = -\frac{p}{\xi_1} \tag{51}$$

Hence

$$f\,'(p) = i\left[r + (z+z_0)\,\frac{p}{\xi_1}\right] \tag{52}$$

Any saddle points will satisfy

$$p_S = -\frac{r}{z+z_0}\,\xi_1 \tag{52}$$

or

$$p_S^2\,[\,(z+z_0)^2 + r^2\,] = \frac{r^2}{\alpha_1^2}\ ; \qquad p_S > 0 \quad (\mathrm{Re}\,(\xi_1) \geq 0) \tag{53}$$

and finally

$$p_S = \frac{1}{\alpha_1}\,\frac{r}{R_0{}'} \tag{54}$$

where $R_0{}' = \sqrt{r^2 + (z+z_0)^2}$ is the distance of the point (r, z) where the wave is computed to the "image" of the source, symmetric of it with respect to the discontinuity.

So, *p_S is just the ray parameter of the reflected ray in classical optics.*

(The – sign in (52) comes from the condition $p_S > 0$ (remember $z, z_0 < 0$) since $\mathrm{Re}\,(\xi_1)$ has to be positive (or null).

THEN,

$$f\,''(p) = i\left[\frac{z+z_0}{\xi_1} + (z+z_0)\,\frac{p^2}{\xi_1^3}\right] = i\,\frac{z+z_0}{\alpha_1^2\,\xi_1^3} \tag{55}$$

which for the value saddle $p = p_S$ takes the form:

$$f\,''(p_S) = i\,\frac{z+z_0}{\alpha_1^2}\left[\frac{-\,r^3}{p_S^3\,(z+z_0)^3}\right] = -\,i\,\frac{r^3}{\alpha_1^2\,(z+z_0)^2\,p_S^3} \tag{56}$$

Substituting now into (56) of Chapter 1, we obtain

$$\phi^{P,\,refl.} = \left(\frac{\omega}{2\pi r}\right)^{1/2} e^{-i\,(\omega t - \pi/4)}\ \acute{P}\grave{P}\,(p_S)\,\frac{p_S^{1/2}}{\xi_{1S}}\,e^{\,i\omega[p_S r - \xi_{1S}(z+z_0)]}\,.\,\sqrt{\frac{2\pi\,p_S^3}{i\omega r^3}\,\alpha_1^2\,(z+z_0)^2} \tag{57}$$

$$= \acute{P}\grave{P}\,(p_S)\cdot e^{\,i\,\omega\left(-t + p_S r + p_S\frac{(z+z_0)^2}{r}\right)}\,.\,\frac{e^{\,i\pi/4}}{\sqrt{i}}\cdot\frac{1}{\xi_{1S}}\cdot\sqrt{\frac{2\pi\,\omega\,\alpha_1^2\,(z+z_0)^2\,p_S^4}{2\pi\,\omega\,r^4}}$$

$$= \acute{P}\grave{P}\,(p_S)\cdot e^{\,i\,\omega\left[-t + \frac{\sqrt{r^2+(z+z_0)^2}}{\alpha_1}\right]}\,.\,\sqrt{\frac{\alpha_1^2\,(z+z_0)^2}{r^4\,\xi_{1S}^2}\cdot p_S^4}$$

Remembering

$$\frac{p_S}{\xi_{1S}} = -\frac{r}{z+z_0} > 0 \qquad \text{(xx)}$$

and $R_0{}' = \sqrt{r^2 + (z+z_0)^2}$,

$$\frac{1}{R_0{}'} = \frac{\alpha_1\, p_S}{r} = \frac{\alpha_1\,(z+z_0)\, p_S r}{(z+z_0)\, r^2} \qquad (58)$$

we finally obtain

$$\phi^{P,\,refl.} = \frac{1}{R_0{}'} \cdot \acute{P}\grave{P}\,(p_S) \cdot e^{i\omega\left(\frac{R_0{}'}{\alpha_1} - t\right)} \qquad (59)$$

NOTE

- $R_0{}'$ is the distance to the image of the source in medium 2;
- p_S is the horizontal slowness of the reflected wave

$$p_S = \frac{1}{\alpha_1}\frac{r}{\sqrt{(z+z_0)^2 + r^2}} \qquad (60)$$

- $(pr - \xi_1 z - \xi_1 z_0) = R_0{}' / \alpha_1$ is the *travel time of the reflected wave.*

ALL IN ALL, THIS IS A SPECTACULAR RESULT:

The potential of the reflected phase is (LOCALLY !!!) that of a spherical wave coming from the symmetric of the source, and weighted by the planar reflection coefficient for the particular slowness p_S at the saddle, i.e., *of the reflected wave.*

BUT, that does not constitute a spherical wave, because that reflection coefficient depends on p_S, and hence on the particular point where the potential is computed.

- p_S has to be smaller than $1/\alpha_1$, but $1/\alpha_2$ is not involved... p_S could be larger than $1/\alpha_2$ if $\alpha_2 > \alpha_1$.

→ **The case of post-critical reflection: $p_S \geq 1 / \alpha_2$**

This very important case describes the so-called *Head Wave* used in seismic refraction experiments, and provides a full quantification of its amplitude.

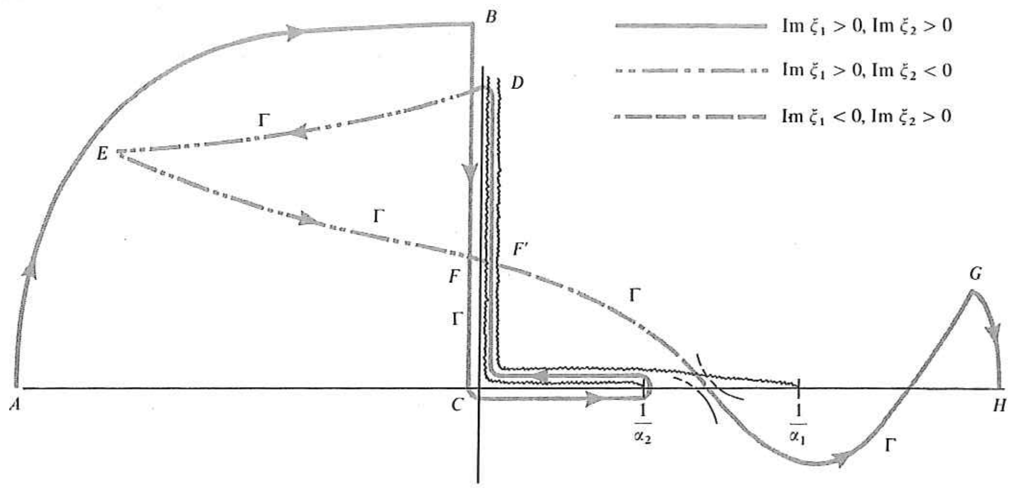


FIGURE **6.9**

The integration path Γ in the complex ray-parameter plane for obtaining P^{refl} (see (6.17)) when $\alpha_1 < \alpha_2$ but $p_s > 1/\alpha_2$. Note that three Riemann sheets are needed as shown at the upper right. Starting at point A, there is no contribution from arc AB. For BC, C to $1/\alpha_2$, and around the cut $\text{Im}\,\xi_2 = 0$, the path stays on the top sheet to D, which is some point sufficiently far up to give a negligible integrand. Crossing the cut at D, the path must descend to $\text{Im}\,\xi_1 > 0$, $\text{Im}\,\xi_2 < 0$ in order to keep the integrand analytic. From E to the saddle point and on to G is exactly the path of steepest descents; at F, the path crosses to $\text{Im}\,\xi_1 < 0$, $\text{Im}\,\xi_2 > 0$, and crosses back to the top sheet at the saddle. A large arc GH contributes nothing.

As shown on the Figure [*Aki & Richards'* (6.9)], in this case the situation is made much more complex by the fact that the path from the saddle dives under the wrong Riemann sheet, which cannot be connected simply to the upper left quadrant of the complex p plane.

Accordingly, the following changes will take place

- The coefficient $\acute{P}\grave{P}$ in (59) will become complex, which will result in a *Phase Shift* of the reflected wave;

- The path of integration Γ has to be deformed around the branch cuts, and **there will now be additional contributions to the integral (46) in the form**

$$\int_{i\infty}^{0} \;;\quad \int_{0}^{1/\alpha_2} \;;\quad \int_{1/\alpha_2}^{0} \;;\quad \int_{0}^{i\infty} \tag{62}$$

In the interval $\{0,\, 1/\alpha_2\}$, we can define the *real* refraction angle $i_2 = \sin^{-1} p\alpha_2$, and ξ_2 will take the value $\cos i_2 / \alpha_2$ below the branch cut, and $-\cos i_2 / \alpha_2$ above it. For large enough ω, the contributions of the integrals along the imaginary axis in (62) are negligible, and the two remaining integrals will be

$$\int_{0}^{1/\alpha_2} + \int_{1/\alpha_2}^{0} = \tag{63}$$

$$= \sqrt{\frac{\omega}{2\pi r}} \cdot e^{-i(\omega t + \frac{\pi}{4})} \cdot \int_{0}^{1/\alpha_2} [\; \acute{P}\grave{P}\,(\cos i_2 > 0) - \acute{P}\grave{P}\,(\cos i_2 < 0)\,] \cdot \frac{\sqrt{p}}{\xi_1} \cdot \exp[\omega\, f(p)] \cdot dp$$

In (63), the function f is given by

$$f(p) \;=\; i\,(pr - \xi_1\, z - \xi_1\, z_0) \tag{51}$$

and the brackets take a form derived from Equation (41R) of Chapter 3:

$$R \;=\; \frac{\rho_2\,\alpha_2\,\cos i_1 \;-\; \rho_1\,\alpha_1\,\cos i_2}{\rho_2\,\alpha_2\,\cos i_1 \;+\; \rho_1\,\alpha_1\,\cos i_2} \tag{41R}$$

NOTE that it is legitimate to use (41R) which was a *displacement* coefficient, even though we need one for *potentials* here, because we reflect in the same medium and without conversion, so the two ratios of displacements to potentials (which are the full wavenumbers $k = \omega\,/\,\alpha_1$) are the same.

$$[\; \acute{P}\grave{P}\,(\cos i_2 > 0) - \acute{P}\grave{P}\,(\cos i_2 < 0)\,] \;= \tag{64}$$

$$= \left[\frac{\rho_2\,\alpha_2\,\cos i_1 \;-\; \rho_1\,\alpha_1\,\cos i_2}{\rho_2\,\alpha_2\,\cos i_1 \;+\; \rho_1\,\alpha_1\,\cos i_2} - \frac{\rho_2\,\alpha_2\,\cos i_1 \;+\; \rho_1\,\alpha_1\,\cos i_2}{\rho_2\,\alpha_2\,\cos i_1 \;-\; \rho_1\,\alpha_1\,\cos i_2}\right] = \frac{-4\;\rho_1\,\rho_2\,\alpha_1\,\alpha_2\cos i_1\,\cos i_2}{\rho_2^2\alpha_2^2\,\cos^2 i_1 \;-\; \rho_1^2\alpha_1^2\,\cos^2 i_2}$$

where $[\cos i_2 \geq 0]$.

We then elect to expand $f(p)$ in the vicinity of $1\,/\,\alpha_2$:

$$f(p) \;=\; f\left(\frac{1}{\alpha_2}\right) + \left(p - \frac{1}{\alpha_2}\right)\cdot f\,'\left(\frac{1}{\alpha_2}\right) \tag{65}$$

since the exponential will oscillate rapidly away from that value. In that vicinity,

$$\cos^2 i_2 \;=\; 1 - \sin^2 i_2 \;=\; \frac{1}{\alpha_2^2} - p^2 \;\approx\; \left(\frac{1}{\alpha_2} - p\right)\cdot\frac{1}{2\,\alpha_2}\;; \qquad \cos i_2 \;=\; \frac{\sqrt{1\,/\,\alpha_2 - p}}{\sqrt{2\,\alpha_2}} \tag{66}$$

In the integral (63), and in the vicinity of $p = 1\,/\,\alpha_2$, the term $\sqrt{p}\,/\,\xi_1$ becomes $[\alpha_1/(\cos i_1\,\sqrt{\alpha_2})]$.

Defining the critical angle i_C through

$$i_C \;=\; \sin^{-1}\frac{\alpha_1}{\alpha_2}\;; \qquad \cos i_C \;=\; \sqrt{1 - \frac{\alpha_1^2}{\alpha_2^2}} \tag{67}$$

we transform (63) into (68)

$$\int_0^{1/\alpha_2} \frac{-4\rho_1\,\rho_2\,\alpha_1\,\alpha_2\,\cos i_1}{\rho_2^2\alpha_2^2\,\cos^2 i_C}\cdot\frac{\sqrt{1\,/\,\alpha_2 - p}}{\sqrt{2\,\alpha_2}}\cdot\frac{\alpha_1}{\cos i_1\,\sqrt{\alpha_2}}\cdot e^{\,\omega\, f(1/\alpha_2)}\cdot\exp\left[\omega(p - 1/\alpha_2)\,f'(1/\alpha_2)\right]\cdot dp \tag{68}$$

$$= \frac{-2\sqrt{2}\;\rho_1\,\alpha_1^2}{\rho_2\,\alpha_2^2\,\cos^2 i_C}\cdot\exp\left[\,\omega\, f\left(\frac{1}{\alpha_2}\right)\right]\cdot\int_0^{\frac{1}{\alpha_2}}\sqrt{\frac{1}{\alpha_2} - p}\cdot\exp\left[\,i\,\omega\, L\left(p - \frac{1}{\alpha_2}\right)\right]\cdot dp$$

where we have further defined

$$L \;=\; r - \frac{1}{\alpha_2}\,\frac{|\,z + z_0\,|}{\xi_1} \tag{69}$$

We now change variables (the end justifies the means...) to

$$p - \frac{1}{\alpha_2} \;=\; i\;y^2\;; \qquad dp \;=\; 2\,i\;y\;dy \tag{70}$$

The integral in (69) becomes

$$-2i\,e^{-i\pi/4}\cdot\int_0^{\sqrt{i/\alpha_2}} y^2\,e^{-\omega L y^2}\cdot dy \tag{71}$$

As $\omega \to \infty$, the upper bound of the integral in (71) can be moved to $\sqrt{i}\,\infty$, and then the integral can be moved onto the real axis, where it becomes

$$\int_0^{\infty} y^2 \exp(-\omega L y^2)\cdot dy \tag{72}$$

Note that

$$\int_0^{\infty} e^{-y^2}\,dy = \frac{\sqrt{\pi}}{2} = \left[\,y\,e^{-y^2}\right]_0^{\infty} + \int_0^{\infty} 2\,y^2\,e^{-y^2}\,dy\;; \quad \text{hence} \quad \int_0^{\infty} y^2\,e^{-y^2}\,dy = \frac{\sqrt{\pi}}{4} \tag{73a}$$

$$\int_0^{\infty} y^2 \exp(-\omega L y^2)\cdot dy = (\omega L)^{-3/2}\cdot\frac{\sqrt{\pi}}{4} \tag{73b}$$

Finally, the full integral substituting for (56) will have the form

$$\phi^{P,\,Refr.} = \sqrt{\frac{\omega}{2\pi r}}\cdot\frac{e^{-i\omega t}\,e^{i\pi/4}\,\sqrt{2}\,i\,\sqrt{\pi}\,e^{-i\pi/4}}{(\omega L)^{3/2}}\cdot\frac{\rho_1\,\alpha_1^2}{\rho_2\,\alpha_2^2\cos^2 i_C}\cdot e^{i\omega\left(\frac{r}{\alpha_2}-\frac{\cos i_1}{\alpha_1}(z+z_0)\right)} \tag{74a}$$

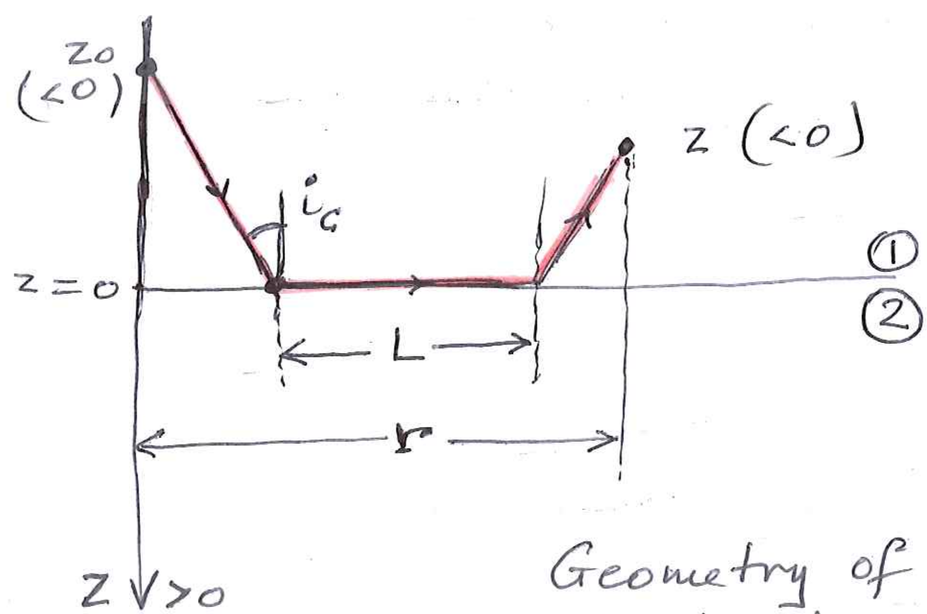
$$\phi^{P,\,Refr.} = \frac{i}{\omega}\;e^{i\omega(t_h-t)}\cdot\frac{1}{\sqrt{r\,L^3}}\cdot\frac{\rho_1}{\rho_2}\,\frac{\alpha_1^2}{\alpha_2^2}\cdot\frac{1}{\cos^2 i_C} \tag{74b}$$

[*Aki and Richards,* (6.25), p. 212)]

where we have introduced the final notation:

$$t_h = \frac{r}{\alpha_2} + \frac{\cos i_C\,|z+z_0|}{\alpha_1} \tag{75}$$

Note the interpretations of L and t_h in the context of the *HEAD (Refracted) WAVE* depicted on the figure:



Geometry of the head wave.

- $L = r - \frac{1}{\alpha_2} \frac{|z+z_0|}{\xi_1}$ is the length of path in medium 2, crawling at grazing incidence below the discontinuity;

- t_h, which can re-written as

$$t_h = \frac{L}{\alpha_2} + \frac{|z+z_0|}{\xi_1 \alpha_2} + \frac{\cos i_C |z+z_0|}{\alpha_1} = \frac{L}{\alpha_2} + \frac{|z+z_0|}{\alpha_1} \left(\cos i_C + \frac{\alpha_1^2}{\alpha_2^2 \cos i_C} \right) \quad (76)$$

$$= \frac{L}{\alpha_2} + \frac{1}{\alpha_1} \frac{|z+z_0|}{\cos i_C}$$

is the travel time of the head wave: a segment L at velocity α_2 plus the two segments $|z+z_0| / \cos i_C$ at velocity α_1.

Finally, note that

- *THERE ARE MANY, MANY APPROXIMATIONS !*

- the dependence on r (or L) is in $1/\sqrt{r L^3} \approx 1/r^2$, which is significantly faster than expected from an intuitively cylindrical wave. However, this is not surprising, since at each point while it is propagating in Medium 2, the head wave leaks back energy into Medium 1.

- The $1/\omega$ frequency dependence. This means an **integration** of the source time function. Everything else being equal, the head wave will be *less high-frequency* than the source.

→ *The high-frequency assumption ($\omega \to \infty$) widely used in this derivation becomes somewhat shaky....*