13 April 2016

Chapter 3

Seismic Plane Waves
(Mostly Reflection and Transmission)

Aki and Richards [1980]; *Chapter 5.*

1. Recalling a few results from Chapter 2, and *Introducing slightly different notations*

- *Slowness vector*

We consider here the case of plane waves, and introduce a **vector slowness s** such that

$$\mathbf{u} = \mathbf{u}(t - \mathbf{s} \cdot \mathbf{x}) \tag{1}$$

which satisfies

$$\mathbf{s} = \frac{1}{c}\, \hat{\imath} \tag{2}$$

where $\hat{\imath}$ is the only direction in space along which derivatives are not identicially zero, and c the velocity of propagation of the wave (α or β). It follows from the fundamental equation of dynamics that

$$\left(\rho - \frac{\mu}{c^2}\right)\ddot{\mathbf{u}} \times \mathbf{s} = \mathbf{0}\,; \qquad \left(\rho - \frac{\lambda + 2\mu}{c^2}\right)\ddot{\mathbf{u}} \cdot \mathbf{s} = 0 \qquad \text{(3\_A\&R 5.3)}$$

Therefore, either ($c^2 = \mu / \rho$) and $\mathbf{u} \cdot \mathbf{s} = 0$ (S wave) or ($c^2 = (\lambda + 2\mu) / \rho$) and $\mathbf{u} \times \mathbf{s} = \mathbf{0}$ (P wave).
For, if c^2 had neither of these two values, then **u** should be both perpendicular and parallel to **s**, and hence **0**.

- *Potentials*

We recall that the displacement u is written as its Helmholtz decomposition;

$$\mathbf{u} = \mathbf{u^P} + \mathbf{u^S} = \mathbf{grad}\,\phi + \mathbf{curl}\,\Psi \qquad \text{with} \qquad \text{div}\,\Psi = 0 \tag{4}$$

Letting

$$\mathbf{u^P} = u(t - \mathbf{s} \cdot \mathbf{x}) \cdot \hat{\imath} \tag{5}$$

it is easily verified that

$$\phi = -u^{-I}(t - \mathbf{s} \cdot \mathbf{x}) \tag{6}$$

where the notation f^{-I} denotes the (−1)th derivative of the function f, namely its *primitive* or first integral.

Similarly, letting

$$\mathbf{u}^S = u(t - \mathbf{s} \cdot \mathbf{x}) \cdot \hat{e}^S \qquad \hat{e}^S \cdot \mathbf{s} = 0 \tag{7}$$

we obtain

$$\Psi = \beta \cdot u^{-1}(t - \mathbf{s} \cdot \mathbf{x}) \cdot (\hat{l} \times \hat{e}^S) \tag{8}$$

Note (i) that the vector potential Ψ is perpendicular to both the S displacement and the slowness vector; and (ii) that the ratio of a potential to its displacement involves not only integration, but also the velocities α or β.

- *Energy density*

  The energy density, per unit volume, is composed of kinetic and potential (elastic) energy. The kinetic energy density is in all cases

$$e_K = \frac{1}{2} \rho (\dot{u})^2 \tag{9}$$

while the density of elastic energy can be written as

$$e_E = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \tag{10}$$

where ε_{ij} and σ_{ij} are the local strain and stress tensor, respectively. In particular

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = -\frac{1}{2}(\dot{u}_i s_j + \dot{u}_j s_i) \tag{11}$$

(NOTE ERROR (Derivative dots Missing) in Aki and Richards [1980; last line of Page 126])

In particular,

$$\varepsilon_{ll} = -\dot{\mathbf{u}} \cdot \mathbf{s} \tag{12}$$

Invoking Hooke's law

$$\sigma_{ij} = \lambda \varepsilon_{ll} \delta_{ij} + 2\mu \varepsilon_{ij} \tag{2\_6}$$

we find

$$e_E = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} = \frac{\lambda}{2} \varepsilon_{ll} \varepsilon_{ii} + \mu \varepsilon_{ij} \varepsilon_{ij} \tag{13}$$

According to (11),

$$\varepsilon_{ll} = -\dot{\mathbf{u}} \cdot \mathbf{s}\,; \qquad \varepsilon_{ll} \varepsilon_{ii} = (\dot{\mathbf{u}} \cdot \mathbf{s})^2 \tag{14}$$

and

$$\varepsilon_{ij} \varepsilon_{ij} = \frac{1}{2}[(\dot{\mathbf{u}} \cdot \mathbf{s})^2 + (\dot{\mathbf{u}})^2 (\mathbf{s})^2] \tag{15}$$

Hence

$$e_E = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} = \frac{1}{2}\left[(\lambda + \mu)(\dot{\mathbf{u}} \cdot \mathbf{s})^2 + \mu (\dot{\mathbf{u}})^2 (\mathbf{s})^2\right] \tag{16}$$

(NOTE ERROR (μ should replace λ) in Aki and Richards [1980; Last Term of (5.4)])

→ For a P wave, for which $\mathbf{u} // \mathbf{s}$, the terms in (16) add up to

$$e_E = \frac{1}{2}(\lambda + 2\mu)(\dot{\mathbf{u}})^2(\mathbf{s})^2 = \frac{1}{2}\frac{\lambda + 2\mu}{\alpha^2}\cdot(\dot{\mathbf{u}})^2 = \frac{1}{2}\rho(\dot{\mathbf{u}})^2 = e_K \qquad (17)$$

→ For an S wave for which $\mathbf{u} \cdot \mathbf{s} = 0$, the only remaining is the last one, leading to

$$e_E = \frac{1}{2}\mu(\dot{\mathbf{u}})^2(\mathbf{s})^2 = \frac{1}{2}\frac{\mu}{\beta^2}\cdot(\dot{\mathbf{u}})^2 = \frac{1}{2}\rho(\dot{\mathbf{u}})^2 = e_K \qquad (18)$$

so that in both cases, we verify the identity of the kinetic and potential energy densities.

2. Plane Waves at Surfaces of Discontinuity

• *Boundary conditions*

We consider here a surface Σ separating two media indexed 1 and 2, featuring different physical properties (ρ, λ, μ). During the passage of a seismic disturbance characterized by fields $\mathbf{u}^1$ and $\mathbf{u}^2$ (and hence by stress fields σ^1 and σ^2), a number of physical conditions must be satisfied.

→ *Conditions on the displacement fields* **u**

If the two media are solid, the interface must remain **welded**, *i.e., the two displacement fields of media 1 and 2 must be the same on both sides of the interface:*

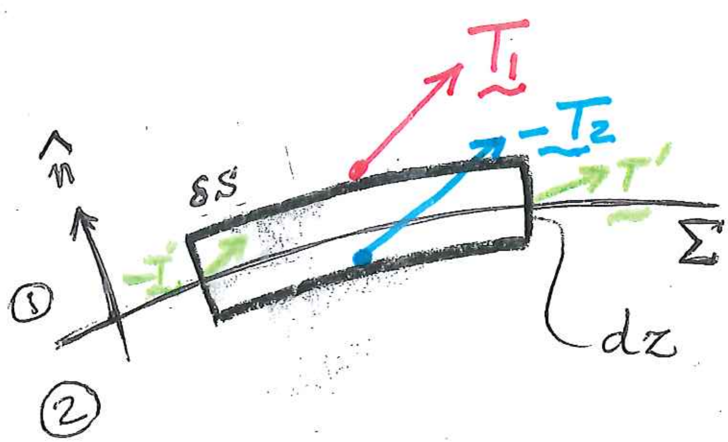
$$\mathbf{u}^1 = \mathbf{u}^2 \tag{19}$$

If at least one medium is **fluid**, it is acceptable to have a discontinuity of displacement in a direction parallel to the surface Σ; in other words, the condition (19) must be satisfied only along the local normal to Σ ("the vertical"), so as to have neither penetration of one medium by the other, nor creation of a vacuum†. Let this direction be, locally, $\hat{\mathbf{e}}_z$; then (19) reduces to

$$u_z^1 = u_z^2 \tag{20}$$

→ *Conditions on the stress fields* $\boldsymbol{\sigma}$

In order to obtain this second set of boundary conditions, we consider a small element of material straddling the discontinuity as shown on Figure 1:



† While the former condition is always upheld, the seond one (no creation of vacuum) may be violated when materials are poorly welded to each other (*e.g.,* sedimentary structures or precarious rock formations). When attacked with vertical accelerations greater than *g*, layered structures can be decomposed (as was the case during the 2011 Christchurch earthquake), and precarious structures flung up.

This chapter will consider seismic displacements of small enough amplitude that such situations are ruled out, and hence that the vertical continuity of displacements is always met.

For this purpose, we consider the surface δS as small, but fixed, and the thickness dz of the element as small *and eventually going to zero* in the mathematical sense of a limit.

If the surface can be regarded as planar (*i.e.,* its radius of curvature is much larger than all the wavelengths involved in the problem), then our element can be thought of as a coin ("penny") of which one half is located in each medium.

* We now write the balance of forces applied to the penny. The force applied to the top face (red on Figure 1) is

$$\mathbf{T}_1 = \boldsymbol{\sigma}^1 \, \hat{\boldsymbol{n}} \cdot \delta S \tag{21t}$$

and that applied to the bottom face (blue on Figure 1)

$$\mathbf{T}_2 = -\boldsymbol{\sigma}^2 \, \hat{\boldsymbol{n}} \cdot \delta S \tag{21b}$$

The minus sign comes from the fact that, on the bottom face, the outgoing normal is $-\hat{\mathbf{n}}$.

The sum of those contributions is

$$\mathbf{T_1} + \mathbf{T_2} = (\sigma_1 - \sigma_2) \, \hat{\boldsymbol{n}} \cdot \delta S \tag{22}$$

which is of order δS.

The contributions from the sides of the penny (schematized in green on Figure 1), will be on the order of the stresses $\boldsymbol{\sigma}$ times the lateral surface of the edge of the penny, *i.e.,* $dz \cdot \sqrt{\delta S}$. When δS is kept a constant, and $dz \to 0$, these lateral contributions go to zero; hence they can be neglected with respect to (22).

Thus, the resultant of forces acting on the penny is simply (22). According to Newton's law ("$\mathbf{f} = m\,\mathbf{a}$"), it will convey an acceleration γ to the penny of mass $dm = \delta S \, dz$:

$$\gamma = \frac{1}{dm} (\sigma_1 - \sigma_2) \, \hat{\boldsymbol{n}} \cdot \delta S = \frac{1}{\bar{\rho} \, dz} (\sigma_1 - \sigma_2) \, \hat{\boldsymbol{n}} \tag{23}$$

(where $\bar{\rho}$ is the average density of the coin), which will become infinite, and hence non-physical as $dz \to 0$, **except if**

$$(\sigma_1 - \sigma_2) \, \hat{\boldsymbol{n}} = 0 \tag{24}$$

which constitutes the second boundary condition, complementing (19). If $\hat{\boldsymbol{n}}$ is in the z direction ($\hat{\boldsymbol{n}} = \hat{\mathbf{e}}_\mathbf{z}$), (24) can be rewritten

$$\sigma^1_{xz} = \sigma^2_{xz} \; ; \qquad \sigma^1_{yz} = \sigma^2_{yz} \; ; \qquad \sigma^1_{zz} = \sigma^2_{zz} \, . \tag{25}$$

Note that when one medium, say 2, is fluid, the shear stresses σ^2_{xz} and σ^2_{yz} are identically zero. In practical terms, the boundary conditions can thus be summarized as:

SOLID – SOLID

$$u^1_x = u^2_x \; ; \qquad u^1_y = u^2_y \; ; \qquad u^1_z = u^2_z \; ; \tag{26ss}$$

$$\sigma^1_{xz} = \sigma^2_{xz} \; ; \qquad \sigma^1_{yz} = \sigma^2_{yz} \; ; \qquad \sigma^1_{zz} = \sigma^2_{zz} \, .$$

SOLID (1) — FLUID (2)

$$u_z^1 = u_z^2 \; ; \qquad (26sf)$$

$$\sigma_{xz}^1 = 0 \; ; \qquad \sigma_{yz}^1 = 0 \; ; \qquad \sigma_{zz}^1 = \sigma_{zz}^2 \; .$$

FLUID — FLUID

$$u_z^1 = u_z^2 \; ; \qquad (26ff)$$

$$\sigma_{zz}^1 = \sigma_{zz}^2 \; .$$

- *Snell's Law*

The derivation of Snell's law is made relatively complex in the case of solids because of the existence of both P and S waves, and of the presence of six boundary conditions. This is why we will first derive its concepts in the simpler case of two liquids.

The Liquid-Liquid Case: Full Derivation

We define the surface as the plane $z = 0$, with negative z in Medium 1, and positive z in Medium 2. We consider the case of a plane wave (hence there exist some vector **s**) incident in Medium 1. By consider it "incident", we tacitly imply that its source is towards the negative z, and because of causality arguments, it must have a positive s_z^1. By the same causality argument, we can exclude any negative s_z^2, because in Medium 2, we are allowed infinite positive values of z, and such a propagation would be non causal.

In both media, we can conduct an *a priori* 4-th order Fourier decomposition in frequency (integrating over time t) and the three components of wave vector (or slowness), integrating over the components of space (x, y, z), and write **u** as

$$\mathbf{u} = \int_{-\infty}^{+\infty} e^{i\omega t} d\omega \int_{-\infty}^{+\infty} e^{-ik_x x} dk_x \int_{-\infty}^{+\infty} e^{-ik_y y} dk_y \int_{-\infty}^{+\infty} e^{-ik_z z} dk_z \cdot \mathbf{U}(\omega, k_x, k_y, k_z) \qquad (28)$$

In Medium 2, the wave equation and the condition of causality will require that

$$k_z^2 = k_z^T = \sqrt{\frac{\omega^2}{\alpha_2^2} - (k_x^2)^2 - (k_y^2)^2} \qquad (29)$$

This will represent a *transmitted wave* (Hence the superscript T).

In medium 1, for any given k_x and k_y, we can have two choices of k_z, because we are bounded by $z < 0$, and so the causality problem does not arise. The positive value of k_z will be represented by the incident wave, but they can exist a negative solution, which will represent a *reflected wave*:

$$k_z^R = -\sqrt{\frac{\omega^2}{\alpha_1^2} - (k_x^1)^2 - (k_y^1)^2} \qquad (30)$$

We now assume that the incident wave is harmonic, *i.e.,* it has only one frequency ω^I. We apply (28) to the vector $\mathbf{u^2} - \mathbf{u^1}$ in (26II), which must apply for all times t; $\mathbf{u^2}$ is made only of $\mathbf{u^T}$; $\mathbf{u^1}$ is a combination of $\mathbf{u^I}$ and $\mathbf{u^R}$. Because of the linearity of the Fourier transform, we find that the reflected and transmitted waves can have only the Fourier component ω^I. Similarly, because (26II) must hold for all x and all y at the surface $z = 0$, the reflected and transmitted waves can only have the same k_x and k_y as the incident wave. *This is caqlled Snell's Law.*

A physical expression of Snell's law is then:

$$\omega^R = \omega^T = \omega_I \qquad (31t)$$

$$k_x^R = k_x^T = k_x^I \qquad (31x)$$

$$k_y^R = k_y^T = k_y^I \qquad (31y)$$

Accordingly, we drop all superscripts on those variables. We can also orient y so that $\mathbf{\hat{e}_y} \cdot \mathbf{k^I} = \mathbf{\hat{e}_y} \cdot \mathbf{\hat{e}_z} = 0$. We are then left with the two equations

$$k_z^R = -k_z^I \qquad (32R)$$

$$k_z^T = \sqrt{\frac{\omega^2}{\alpha_2^2} - (k_x)^2} \qquad (32T)$$

$$k_z^R = -\sqrt{\frac{\omega^2}{\alpha_1^2} - (k_x)^2} \qquad (32R)$$

or, introducing the incidence angles i^I, i^R and i^T, such that

$$k_z^I = \frac{\omega}{\alpha_1}\cos i^I \ ; \qquad k_x = \frac{\omega}{\alpha_1}\sin i^I \qquad (34I)$$

$$k_z^I = -\frac{\omega}{\alpha_1}\cos i^R \ ; \qquad k_x = \frac{\omega}{\alpha_1}\sin i^R \qquad (34R)$$

$$k_z^T = \frac{\omega}{\alpha_2}\cos i^T \ ; \qquad k_x = \frac{\omega}{\alpha_2}\sin i^T \qquad (34T)$$

we obtain the classical relations:

$$\frac{\sin i^I}{\alpha_1} = \frac{\sin i^R}{\alpha_1} = \frac{\sin i^T}{\alpha_2} \qquad (35)$$

WITH THIS RESULT OBTAINED

We can now proceed to try to solve for the amplitudes of the reflected and transmitted waves: Changing notations slightly, we consider an incident wave derived from the potential

$$\phi^I = \phi_0 \exp i\omega(t - s_x x) \cdot \exp\left[-i\omega \frac{\cos i_1}{\alpha_1} z\right] \tag{36I}$$

and similarly

$$\phi^R = R \cdot \phi_0 \exp i\omega(t - s_x x) \cdot \exp\left[i\omega \frac{\cos i_I}{\alpha_1} z\right] \tag{36R}$$

$$\phi^T = T \cdot \phi_0 \exp i\omega(t - s_x x) \cdot \exp\left[-i\omega \frac{\cos i_2}{\alpha_2} z\right] \tag{36T}$$

from which we derive (for $z = 0$) the displacements

$$u_x^I = -i\omega s_x \phi_0 \exp i\omega(t - s_x x) \tag{37Ix}$$

$$u_z^I = -i\omega \frac{\cos i_1}{\alpha_1} \phi_0 \exp i\omega(t - s_x x) \tag{37Iz}$$

$$u_x^R = -i\omega s_x R \phi_0 \exp i\omega(t - s_x x) \tag{37Rx}$$

$$u_z^R = i\omega \frac{\cos i_1}{\alpha_1} R \phi_0 \exp i\omega(t - s_x x) \tag{37Rz}$$

$$u_x^T = -i\omega s_x T \phi_0 \exp i\omega(t - s_x x) \tag{37Tx}$$

$$u_z^T = -i\omega \frac{\cos i_2}{\alpha_2} T \phi_0 \exp i\omega(t - s_x x) \tag{37Tz}$$

the strains

$$\varepsilon_{ll}^I = -\frac{\omega^2}{\alpha_1^2} \phi_0 \exp i\omega(t - s_x x) \tag{38I}$$

$$\varepsilon_{ll}^R = -\frac{\omega^2}{\alpha_1^2} R \phi_0 \exp i\omega(t - s_x x) \tag{38R}$$

$$\varepsilon_{ll}^T = -\frac{\omega^2}{\alpha_2^2} T \phi_0 \exp i\omega(t - s_x x) \tag{38T}$$

and the stresses

$$\sigma_{zz}^I = \lambda_1 \varepsilon_{ll}^I = -\omega^2 \frac{\lambda_1}{\alpha_1^2} \phi_0 \exp i\omega(t - s_x x) \tag{39I}$$

$$\sigma_{zz}^R = \lambda_1 \varepsilon_{ll}^R = -\omega^2 \frac{\lambda_1}{\alpha_1^2} R \phi_0 \exp i\omega(t - s_x x) \tag{39R}$$

$$\sigma_{zz}^T = \lambda_2 \varepsilon_{ll}^T = -\omega^2 \frac{\lambda_2}{\alpha_2^2} T \phi_0 \exp i\omega(t - s_x x) \tag{39T}$$

Equations (26ff) lead to:

$$(1 - R) \frac{\cos i_1}{\alpha_1} = T \frac{\cos i_2}{\alpha_2} \tag{40d}$$

$$(1 + R)\,\rho_1 \;=\; T\,\rho_2 \tag{40s}$$

whose solution is

$$R \;=\; \frac{\rho_2\,\alpha_2\,\cos i_1 \;-\; \rho_1\,\alpha_1\,\cos i_2}{\rho_2\,\alpha_2\,\cos i_1 \;+\; \rho_1\,\alpha_1\,\cos i_2} \tag{41R}$$

$$T \;=\; \frac{2\,\rho_1\,\alpha_2\,\cos i_1}{\rho_2\,\alpha_2\,\cos i_1 \;+\; \rho_1\,\alpha_1\,\cos i_2} \tag{41T}$$

NOTE THAT THESE COEFFICIENTS RELATE TO THE POTENTIALS ϕ^R and ϕ^T.

If we had looked for coefficients expressed as a function of displacements, they would be scaled by $1/\alpha$: In this context, R would be unchanged, but T would be multiplied by α_1 / α_2.

When using published formulæ for reflection/transmission coefficients, it is crucial to always check whether they relate to potentials or displacements.

THE ROAD TO P–SV COUPLING

→ We will now consider expanding these results to the more general case of a solid-solid interface. It is known traditionally that this must involve $P \rightarrow S$ (or $S \rightarrow P$) conversions (except in special cases such as vertical incidence). This may become paradoxical, and so we will provide a justification by examining the horizontal displacements at the boundary of two liquids, as solved above.

Once the coefficients R and T have been obtained, it is possible to go back to Equations (37*x), and to show that the combined *horizontal* displacement u_x in Medium 1 is distinct from that of the transmitted wave in Medium 2. This is made possible because the media are fluid, and thus can slide on top of each other. But, had they been solid, the welded boundary condition (26ss) could not have been met (not to say anything of the condition on the shear stresses σ_{zx}). Hence the necessity to bring in two additional degrees of freedom by involving converted S waves in Media 1 and 2.

A more mathematical argument would be that, once a proper y direction has been selected, (26ss) requires solving four equations, and thus four unknowns (the amplitudes of the reflected P and S waves, and of the transmitted P and S) are necessary.

HOMEWORK 4

- Compute this effect fully. Show that there is no horizontal slushing of the media on top of each other at vertical incidence. Any simple explanation?

THE MOST GENERAL CASE: SOLID–SOLID

We follow here the derivation of *Aki & Richards* [1980; pp. 144–151].

- Medium 1 is taken as "on top" of Medium 2
- The orientation conventions are those of Figure 5.8 (reproduced on Page 11)

- The notation is to use use an acute accent (´) for waves propagating upwards and a grave accent (`) for waves propagating downwards. Thus a term like $\acute{P}\grave{P}$ will mean the reflection coefficient of a P wave incident in Medium 2 (lower half-space) and $\grave{P}\grave{S}$ would be the transmission coefficient of a P wave incident in Medium 1 and converted to an S wave in Medium 2.
- **These coefficients are displacement coefficients**

- *Strategy to compute the coefficients.*

  Consider, for example, a P wave incident in medium 1. It is going down, so we can write its amplitude as $\grave{P}_1$. It creates displacements and stresses at the boundary which are linear in its amplitude $\grave{P}_1$. Similarly, the other 7 possible waves (incident or emerging P or S in each of the two media) create similar terms, and the boundary conditions can be written as A&R's (5.33) p. 145:

$$
\begin{aligned}
\sin i_1(\grave{P}_1 + \acute{P}_1) + \cos j_1(\grave{S}_1 + \acute{S}_1) &= \sin i_2(\grave{P}_2 + \acute{P}_2) + \cos j_2(\grave{S}_2 + \acute{S}_2), \\
\cos i_1(\grave{P}_1 - \acute{P}_1) - \sin j_1(\grave{S}_1 - \acute{S}_1) &= \cos i_2(\grave{P}_2 - \acute{P}_2) - \sin j_2(\grave{S}_2 - \acute{S}_2), \\
2\rho_1\beta_1^2 p \cos i_1(\grave{P}_1 - \acute{P}_1) + \rho_1\beta_1(1 - 2\beta_1^2p^2)(\grave{S}_1 - \acute{S}_1) & \\
= 2\rho_2\beta_2^2 p \cos i_2(\grave{P}_2 - \acute{P}_2) &+ \rho_2\beta_2(1 - 2\beta_2^2p^2)(\grave{S}_2 - \acute{S}_2), \\
\rho_1\alpha_1(1 - 2\beta_1^2p^2)(\grave{P}_1 + \acute{P}_1) - 2\rho_1\beta_1^2 p \cos j_1(\grave{S}_1 + \acute{S}_1) & \\
= \rho_2\alpha_2(1 - 2\beta_2^2p^2)(\grave{P}_2 + \acute{P}_2) &- 2\rho_2\beta_2^2 p \cos j_2(\grave{S}_2 + \acute{S}_2),
\end{aligned}
\tag{5.33}
$$

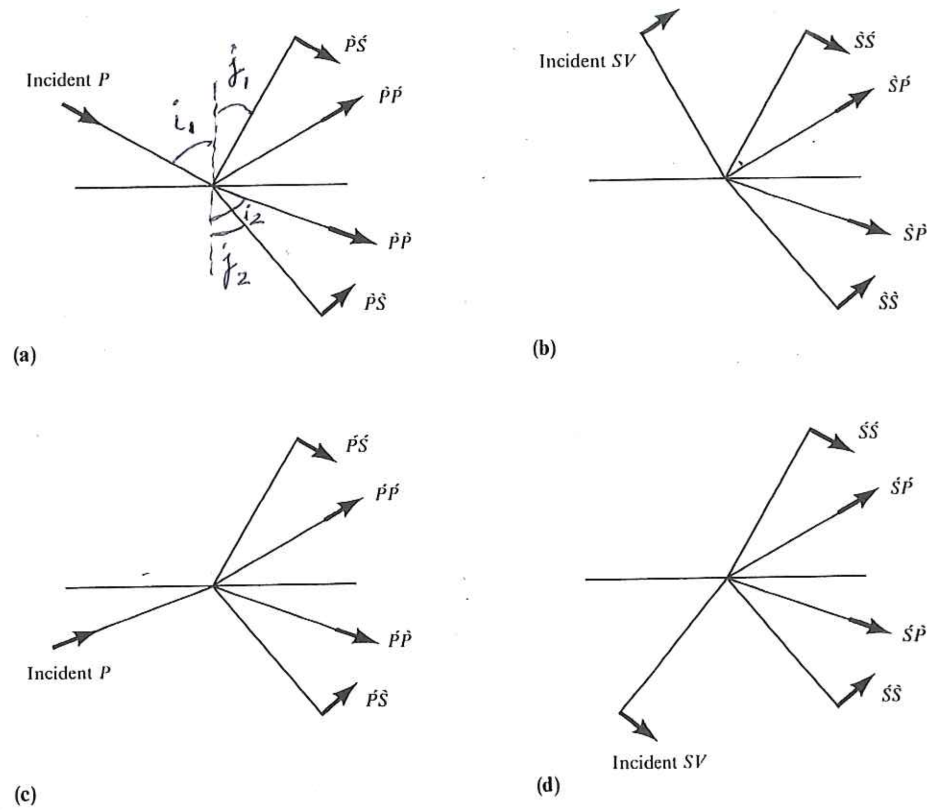


FIGURE **5.8**
Notation for the sixteen possible reflection/transmission coefficients arising for problems of *P-SV* waves at the welded interface between two different solid half-spaces.

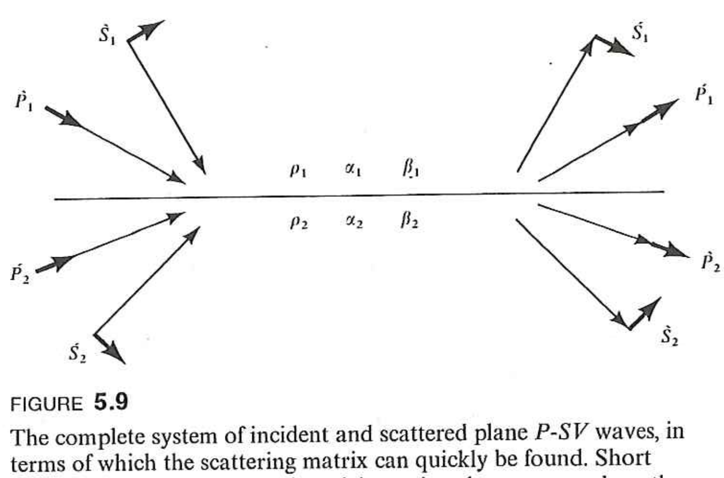


FIGURE **5.9**
The complete system of incident and scattered plane *P-SV* waves, in terms of which the scattering matrix can quickly be found. Short arrows show the direction of particle motion; long arrows show the direction of propagation.

This equation (A&R 5.33) can be rewritten by regrouping incident waves on the right and emerging ones on the left. Because of the linearity of the whole problem, there exist two 4×4 matrices, **M** and **N** such that:

$$\mathbf{M}\begin{pmatrix} \acute{P}_1 \\ \acute{S}_1 \\ \grave{P}_2 \\ \grave{S}_2 \end{pmatrix} = \mathbf{N}\begin{pmatrix} \grave{P}_1 \\ \grave{S}_1 \\ \acute{P}_2 \\ \acute{S}_2 \end{pmatrix}, \qquad (5.34)$$

→ The reflection and transmission coefficients correspond to the amplitudes of the emerging waves when the incident wave is one of the unit vectors. Therefore, they are just the individual elements of the matrix

$$\mathbf{M}^{-1} \cdot \mathbf{N} \qquad (42)$$

In short *(sic!)*, the 16 possible coefficients are given in A&R (5.39, pp. 150–151) which is reproduced on Page 13, using the following notation:

$$p = \frac{\sin i_1}{\alpha_1} = \frac{\sin i_2}{\alpha_2} = \frac{\sin j_1}{\beta_1} = \frac{\sin j_2}{\beta_2} \qquad (43)$$

$$a = \rho_2(1 - 2\beta_2^2 p^2) - \rho_1(1 - 2\beta_1^2 p^2), \qquad b = \rho_2(1 - 2\beta_2^2 p^2) + 2\rho_1\beta_1^2 p^2,$$

$$c = \rho_1(1 - 2\beta_1^2 p^2) + 2\rho_2\beta_2^2 p^2, \qquad d = 2(\rho_2\beta_2^2 - \rho_1\beta_1^2),$$

$$E = b\frac{\cos i_1}{\alpha_1} + c\frac{\cos i_2}{\alpha_2}, \qquad F = b\frac{\cos j_1}{\beta_1} + c\frac{\cos j_2}{\beta_2},$$

$$G = a - d\frac{\cos i_1}{\alpha_1}\frac{\cos j_2}{\beta_2}, \qquad H = a - d\frac{\cos i_2}{\alpha_2}\frac{\cos j_1}{\beta_1},$$

$$D = EF + GHp^2 = (\det \mathbf{M})/(\alpha_1\alpha_2\beta_1\beta_2). \qquad (5.38)$$

$$
\begin{aligned}
\grave{P}\acute{P} &= \left[\left(b\frac{\cos i_1}{\alpha_1} - c\frac{\cos i_2}{\alpha_2}\right)F - \left(a + d\frac{\cos i_1}{\alpha_1}\frac{\cos j_2}{\beta_2}\right)Hp^2\right]\Big/\mathrm{D},\\
\grave{P}\acute{S} &= -2\frac{\cos i_1}{\alpha_1}\left(ab + cd\frac{\cos i_2}{\alpha_2}\frac{\cos j_2}{\beta_2}\right)p\alpha_1/(\beta_1\mathrm{D}),\\
\grave{P}\grave{P} &= 2\rho_1\frac{\cos i_1}{\alpha_1}F\alpha_1/(\alpha_2\mathrm{D}),\\
\grave{P}\grave{S} &= 2\rho_1\frac{\cos i_1}{\alpha_1}Hp\alpha_1/(\beta_2\mathrm{D}),\\
\grave{S}\acute{P} &= -2\frac{\cos j_1}{\beta_1}\left(ab + cd\frac{\cos i_2}{\alpha_2}\frac{\cos j_2}{\beta_2}\right)p\beta_1/(\alpha_1\mathrm{D}),\\
\grave{S}\acute{S} &= -\left[\left(b\frac{\cos j_1}{\beta_1} - c\frac{\cos j_2}{\beta_2}\right)E - \left(a + d\frac{\cos i_2}{\alpha_2}\frac{\cos j_1}{\beta_1}\right)Gp^2\right]\Big/\mathrm{D},\\
\grave{S}\grave{P} &= -2\rho_1\frac{\cos j_1}{\beta_1}Gp\beta_1/(\alpha_2\mathrm{D}),\\
\grave{S}\grave{S} &= 2\rho_1\frac{\cos j_1}{\beta_1}E\beta_1/(\beta_2\mathrm{D}),\\
\acute{P}\acute{P} &= 2\rho_2\frac{\cos i_2}{\alpha_2}F\alpha_2/(\alpha_1\mathrm{D}),\\
\acute{P}\acute{S} &= -2\rho_2\frac{\cos i_2}{\alpha_2}Gp\alpha_2/(\beta_1\mathrm{D}),\\
\acute{P}\grave{P} &= -\left[\left(b\frac{\cos i_1}{\alpha_1} - c\frac{\cos i_2}{\alpha_2}\right)F + \left(a + d\frac{\cos i_2}{\alpha_2}\frac{\cos j_1}{\beta_1}\right)Gp^2\right]\Big/\mathrm{D},\\
\acute{P}\grave{S} &= 2\frac{\cos i_2}{\alpha_2}\left(ac + bd\frac{\cos i_1}{\alpha_1}\frac{\cos j_1}{\beta_1}\right)p\alpha_2/(\beta_2\mathrm{D}),\\
\acute{S}\acute{P} &= 2\rho_2\frac{\cos j_2}{\beta_2}Hp\beta_2/(\alpha_1\mathrm{D}),\\
\acute{S}\acute{S} &= 2\rho_2\frac{\cos j_2}{\beta_2}E\beta_2/(\beta_1\mathrm{D}),\\
\acute{S}\grave{P} &= 2\frac{\cos j_2}{\beta_2}\left(ac + bd\frac{\cos i_1}{\alpha_1}\frac{\cos j_1}{\beta_1}\right)p\beta_2/(\alpha_2\mathrm{D}),\\
\acute{S}\grave{S} &= \left[\left(b\frac{\cos j_1}{\beta_1} - c\frac{\cos j_2}{\beta_2}\right)E + \left(a + d\frac{\cos i_1}{\alpha_1}\frac{\cos j_2}{\beta_2}\right)Hp^2\right]\Big/\mathrm{D}. \qquad (5.39)
\end{aligned}
$$

- In addition to these formaulæ, one must consider the case of *SH* waves, *i.e., S* waves polarized perpendicular to the plane of incidence. In this particular case, the problem is entirely decoupled, *i.e.,* only *SH* waves are transmitted or reflected and the boundary conditions reduce to continuity of u_y and σ_{yz}. As given by *Aki & Richards* [1980, (5.32) p. 144], the corresponding **displacement** coefficients are:

$$\grave{S}\acute{S} = \frac{\rho_1 \beta_1 \cos j_1 - \rho_2 \beta_2 \cos j_2}{\rho_1 \beta_1 \cos j_1 + \rho_2 \beta_2 \cos j_2} = -\acute{S}\grave{S}$$

$$\acute{S}\acute{S} = \frac{2 \rho_2 \beta_2 \cos j_2}{\rho_1 \beta_1 \cos j_1 + \rho_2 \beta_2 \cos j_2} \qquad (44)$$

$$\grave{S}\grave{S} = \frac{2 \rho_1 \beta_1 \cos j_1}{\rho_1 \beta_1 \cos j_1 + \rho_2 \beta_2 \cos j_2}$$

→ There are also special formulæ applicable to the case of a solid-liquid interface (*e.g.,* the core-mantle boundary), and reflection at a free surface. They can be found in a variety of textbooks [*Aki and Richards,* 1980; *Ben-Menahem and Singh,* 1981; *Lay and Wallace,* 1995].

* Such formulæ are known as *Zöppritz'* coefficients. Unfortunately, they are plagued by many misprints in many texts. It appears that those listed in *Aki and Richards* are free of errors, but this is not guaranteed...

* In addition to the warning about potential as opposed to displacement coefficients, note that the orientation conventions shown on Page 11 are not always followed by other authors, *especially regarding the orientation of SV.*

* Finally, some authors have pushed their desire for ambiguity as far as using grazing angles $e_k = \frac{\pi}{2} - i_k$ instead of incidence angles $i_k \cdots$

HOMEWORK 5

- Program the underside reflection coefficient $\acute{P}\grave{P}$ for the 660–km discontinuity. Use a jump of 4.07 to 4.36 g/cm^3 for ρ, 10.25 to 10.64 km/s for α, and 5.61 to 5.90 km/s for β.

(i) Use Formulæ (A&R5.38; Page 13), and find the maximum of $\acute{P}\grave{P}$ as a function of i (or slowness p).

(ii) Comment with respect to efforts towards finding precusrsors to phases such as PKPPKP.

(iii) Compare your results to those obtained by using Formulæ A&R (5.44) p. 153. Discuss.

(iv) *Easier!* Use (44) for a similar study of underside reflection of *SH* waves. Conclusion.

→ *Another important special case:* **Reflection at a free surface**

At a free surface (*e.g.,* the Earth's surface), one of the media is a vacuum, which can be arbitrarily displaced without producing any stresses or restoring forces (both its elastic constants are zero). Thus, there remain only two boundary conditions for the problem of an incident P or SV wave

$$\sigma_{zz}^{1} = 0 \qquad \text{and} \qquad \sigma_{zx}^{1} = 0 \tag{45}$$

and just one for an incident SH wave

$$\sigma_{zy}^{1} = 0 \,. \tag{46}$$

Because (45) involves two boundary conditions, it cannot be satisfied by a single reflected wave and so, an incident P (resp. SV) wave will give rise to a standard reflected P (resp. SV) wave, but also to a *converted refelected* SV (resp. P) wave. The relevant coeffcients are given by *Aki & Richards* [1980, (5.26), (5.27), (5.30), (5.31), p. 140], which are copied below:

(47)

$$\acute{P}\grave{P} = \frac{-\left(\frac{1}{\beta^2} - 2p^2\right)^2 + 4p^2\frac{\cos i}{\alpha}\frac{\cos j}{\beta}}{+\left(\frac{1}{\beta^2} - 2p^2\right)^2 + 4p^2\frac{\cos i}{\alpha}\frac{\cos j}{\beta}}, \tag{5.26}$$

$$\acute{P}\grave{S} = \frac{4\frac{\alpha}{\beta}p\frac{\cos i}{\alpha}\left(\frac{1}{\beta^2} - 2p^2\right)}{\left(\frac{1}{\beta^2} - 2p^2\right)^2 + 4p^2\frac{\cos i}{\alpha}\frac{\cos j}{\beta}}. \tag{5.27}$$

$$\acute{S}\grave{P} = \frac{4\frac{\beta}{\alpha}p\frac{\cos j}{\beta}\left(\frac{1}{\beta^2} - 2p^2\right)}{\left(\frac{1}{\beta^2} - 2p^2\right)^2 + 4p^2\frac{\cos i}{\alpha}\frac{\cos j}{\beta}}, \tag{5.30}$$

$$\acute{S}\grave{S} = \frac{\left(\frac{1}{\beta^2} - 2p^2\right)^2 - 4p^2\frac{\cos i}{\alpha}\frac{\cos j}{\beta}}{\left(\frac{1}{\beta^2} - 2p^2\right)^2 + 4p^2\frac{\cos i}{\alpha}\frac{\cos j}{\beta}}. \tag{5.31}$$

where p is the common ray parameter or horizontal slowness of all the waves:

$$p = \frac{\sin i}{\alpha} = \frac{\sin j}{\beta} = \frac{k_x}{\omega} \tag{48}$$

In addition, for an SH incident wave, the reflection coefficient is identically 1:

$$\acute{S}_H\grave{S}_H = 1 \tag{49}$$

There are several important consequences to the system of Equations (47) (and 49).

- First, when a teleseismic P wave reaches a distant station, *a fortiori* located on the surface of the Earth, it gives rise to a vertical displacement which is not just its own, but the combination of the displacements induced by the incident, reflected P and converted SV waves. Let us assume we operate a *vertical* seismometer and the P wave is incident with amplitude A at an incidence angle i. If we use the orientation conventions of *Aki & Richards* [1980] (reproduced on Page 11), this combined vertical displacement will be

$$A \cdot [\cos i - \acute{P}\grave{P} \cos i + \acute{P}\grave{S} \sin j] \quad (50)$$

rather than the intuitive value $A \cdot \cos i$. This corresponds to a *"response coefficient"*

$$C^P(i_0) = [1 - \acute{P}\grave{P}(i_0)] \cos i_0 + \acute{P}\grave{S}(i_0) \sin j_0 = \frac{\frac{2}{\beta^2}\left(\frac{1}{\beta^2} - 2p^2\right)\cos i_0}{\left(\frac{1}{\beta^2} - 2p^2\right)^2 + \frac{4p^2 \cos i_0 \cos j_0}{\alpha\beta}} \quad (51)$$

Similar expressions can be obtained for the radial (horizontal away from the source) amplitude resulting from the incidence of an SV wave, and for the cross terms (*e.g.,* the horizontal motion resulting from an incident P wave). They can be found in *Okal* [1992, *Seismol. Res. Letts.,* **63,** 169–180, 1992] from which "Figure 4" (Page 17) is taken. The case of an incident SH is very simple: $C^{SH} = 2$ regardless of incidence.

- *NOTE FINALLY* that the common denominator of the four coefficients in (47),

$$\Delta = \left[\frac{1}{\beta^2} - 2p^2\right]^2 + 4p^2 \frac{\cos i}{\alpha} \frac{\cos j}{\beta} \quad (52)$$

has the good fortune of being *strictly* positive, (as long of course as the angles i and j are real), since in the geometry involved i and j cannot be greater than $\pi/2$ and the second term cannot be negative, while the first is also positive; this means that the coefficients (51) cannot blow up (become infinite); big sigh of relief...

Actually, there could be the possibility of an S incidence at exactly $\pi/4$, and a P grazing incidence ($i = \pi/2$), but in this case all terms in both numerators and denominators of (51) vanish. A careful Taylor expansion of these coefficients show than the converted ones ($\acute{P}\grave{S}$ and $\acute{P}\grave{S}$) go to zero, while $\acute{P}\grave{P} \to 1$ and $\acute{S}\grave{S} \to -1$ which saves the day: the coefficients do not blow up.

Incidentally, this situation would require a medium with $\alpha / \beta = \sqrt{2}$, or $\lambda = 0$. This is, to my knowledge, the only physical result directly involving the 1st Lamé coefficient λ.

BUT, IF WE WERE TO ALLOW COMPLEX VALUES OF THE ANGLES i AND j, IT MAY BE A DIFFERENT STORY.... **STAY TUNED !**

Okal

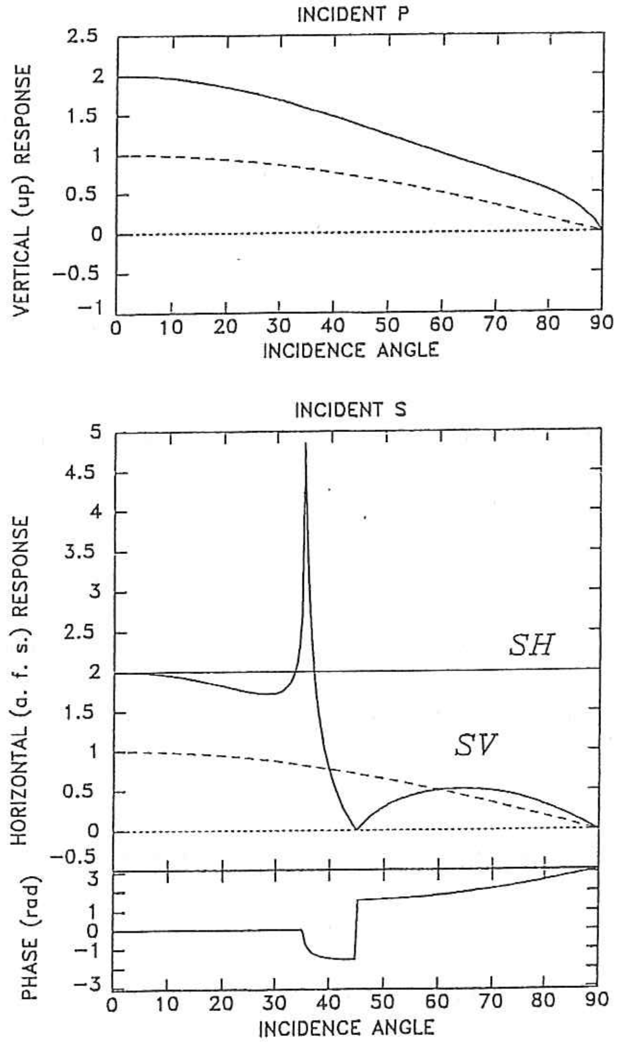


Fig. 4. Surface response coefficients $C^P(i_0)$ and $C^S(j_0)$ as a function of the incidence angle and in the case of a Poisson solid, as described by Equations (18) and (19). *Top:* Response of a vertical seismometer to an incident P wave of unit amplitude. The solid line represents C^P, the dashed one ($\cos i_0$) the contribution of the incident P alone. *Bottom:* Response of a horizontal seismometer polarized away from the source, to an incident SV wave of unit amplitude. The solid curves show the amplitude (top frame) and phase (bottom frame) of C^{SV}. Note that due to critical $S \rightarrow P$ reflection, this coefficient becomes complex around $j_0 = 35°$. C^{SV} is also compared to the surface response for SH waves (identically equal to 2), and to the contribution of the incident wave alone ($\cos j_0$; dashed line). This figure shows that the apparent SH/SV ratio is strongly distorted beyond 30° of incidence.

THE ROAD TO RAYLEIGH WAVES

The so-called inhomogeneous waves

When performing the Fourier transforms (28) in a medium extending infinitely in the x and y directions, the corresponding wavevector components k_x and k_y must remain real, so that the relevant exponentials do not blow up. Assuming a choice of k_x and k_y, k_z is then determined from the wave equation

$$k_z^2 = \frac{\omega^2}{(\alpha^2 \text{ or } \beta^2)} - k_x^2 - k_y^2 \qquad (53)$$

the choice of velocity (α or β) depending on the nature of the wave (P or S).

A problem develops when k_x and k_y are too large, and the solution to (53) would be imaginary. This cannot be tolerated if the medium is unbounded and the coordinate z which multiplies $-i\,k_z$ in the argument of the exponential is allowed to go to plus and minus infinity. ***BUT***, if the medium is bounded by a flat layer (say $z > 0$), then pure imaginary values of $k_z = -i\kappa$ with $\kappa > 0$ are allowed since the exponential $\exp(-i\,k_z\,z)$ would then blow up only for $z \to -\infty$ which is not allowed inside the half-space.

Such *"inhomogeneous"* waves (so named because their amplitude of oscillation is not constant in space...) are legitimate to consider as solutions of the wave equations in a half-space (or a layer).

AND BECAUSE OF THE PROPERTIES OF COMPLEX NUMBERS, ALL THE PREVIOUS ALGEBRA REMAINS CORRECT WITH INHOMOGENEOUS WAVES AND IN PARTICULAR THE REFLECTION/TRANSMISSION/CONVERSION COEFFICIENTS WILL REMAIN EXACT, AS LONG AS WE ALLOW FOR COMPLEX ANGLES OF INCIDENCE FOR WHICH

$$\sin i > 1\,; \qquad \cos i \ \text{ pure imaginary} \qquad (54)$$

We start with the simplest idea, which is to see if a single inhomogenous wave can exist by itself in a half-space. Assume no dependence on y and the case of a P wave. While a wave such as

$$\mathbf{u} = A_P\,\hat{\imath} \cdot \exp\left[\,i(\omega t - k_x\,x) - \kappa\,z\,\right] \qquad (55)$$

is legitimate as long as

$$k_x^2 - \kappa^2 = \frac{\omega^2}{\alpha^2}\,; \qquad \kappa > 0 \qquad (56)$$

the wave must still satisfy the ***two*** boundary conditions (45). The only solution is $A_P = 0$.

However, if we allow both P and S waves, we will have two degrees of freedom (their amplitudes), and so we may be able to find a case where two boundary conditions can be satisfied with a non-trivial solution (*i.e.,* both amplitudes are non-zero). This will be the case if the system relating the boundary conditions to the amplitudes of the waves becomes *degenerate*, *i.e.,* its determinant vanishes. In other words, these two legitimate P and S inhomogeneous waves would just happen to combine to satisfy the two boundary conditions by themselves, without the need of the "reflected" ones which would be illegitimate, since they would have negative κ. That would require a combination of amplitudes A_P and A_S such that

$$A_P \; \acute{P}\grave{P} \; + \; A_S \; \acute{S}\grave{P} \; = \; A_P \; \acute{P}\grave{S} \; + \; A_S \; \acute{S}\grave{S} \; = \; 0 \tag{57}$$

Since the terms $\acute{P}\grave{P}$... are the elements of the matrix $\mathbf{M}^{-1} \cdot \mathbf{N}$ (42), this requires that the matrix be singular, *i.e.,* that its determinant vanish.

Another way of looking at the same problem is to consider that the legitimate waves ($\kappa > 0$) are the reflected $\grave{P}$ and $\grave{S}$ waves of an incident $\acute{P}$ (or/and $\acute{S}$), which would be illegitimate ($\kappa < 0$), except if by a stroke of luck its amplitude is zero. This corresponds to the case when the reflection/conversion coefficients (51) will become infinite, *i.e.,* when the denominator

$$\Delta \; = \left[\frac{1}{\beta^2} - 2\,p^2 \right]^2 \; + \; 4\,p^2 \, \frac{\cos i}{\alpha} \, \frac{\cos j}{\beta} \tag{52}$$

vanishes.

→ Thus, we now attack the problem of trying to make the determinant (52) zero:

$$\left[\frac{1}{\beta^2} - 2\,p^2 \right]^2 \; + \; 4\,p^2 \, \frac{\cos i}{\alpha} \, \frac{\cos j}{\beta} \; = \; 0 \tag{58}$$

- Recall that with i and j both real, hence with real values of the cosines, there is no hope.
- Remember that $\sin i$ and $\sin j$ must remain real (albeit possibly greater than 1) so that amplitudes do not blow up for $x \rightarrow \pm\infty$. Also, Snell's law must be verified. Hence, the first term in (58) is real positive. The only possibility is then to have *both* $\cos i$ *and* $\cos j$ imaginary. This will be the case if both $\sin i$ and $\sin j$ are greater than 1 (which requires only the latter since $\alpha > \beta$):

$$p \; = \; \frac{\sin i}{\alpha} \; = \; \frac{\sin j}{\beta} \; > \; \frac{1}{\beta} \; > \; \frac{1}{\alpha} \tag{59}$$

Equation (58) can be rewritten as a function of p only:

$$\cos i \; = \; i \sqrt{\sin^2 i - 1}\,; \qquad\qquad \cos j \; = \; i \sqrt{\sin^2 j - 1} \tag{60}$$

(the signs of the roots are determined by the condition $\kappa > 0$), and then

$$\left[\frac{1}{\beta^2} - 2\,p^2 \right]^2 \; = \; \frac{4\,p^2}{\alpha\,\beta} \sqrt{(p^2\,\alpha^2 - 1)\,(p^2\,\beta^2 - 1)} \tag{61}$$

leading to

$$\left[\frac{1}{\beta^2} - 2\,p^2 \right]^4 \; = \; \frac{16\,p^4}{\alpha^2\,\beta^2} \, (p^2\,\alpha^2 - 1)\,(p^2\,\beta^2 - 1) \tag{62}$$

and in turn to

$$\frac{1}{\beta^8} - 8\frac{p^2}{\beta^6} + 24\frac{p^4}{\beta^4} - 32\frac{p^6}{\beta^2} + 16\,p^8 = 16\,p^8 - 16\frac{p^6}{\alpha^2} - 16\frac{p^6}{\beta^2} + 16\frac{p^4}{\alpha^2\,\beta^2} \qquad (63a)$$

$$1 - 8\,\beta^2 p^2 + \beta^4 p^4\left(24 - 16\frac{\beta^2}{\alpha^2}\right) - \beta^6 p^6\left(16 - 16\frac{\beta^2}{\alpha^2}\right) = 0 \qquad (63b)$$

Setting ($y = \beta^2\,p^2$) *which must remain greater than 1 (59),* and further considering only a Poisson solid for which $\alpha^2 / \beta^2 = 3$,

$$1 - 8\,y + y^2\left(24 - \frac{16}{3}\right) - y^3\left(16 - \frac{16}{3}\right) = 0 \qquad (64a)$$

$$1 - 8\,y + \frac{56}{3}\,y^2 - \frac{32}{3}\,y^3 = 0 \qquad (64b)$$

which is the classical equation for the phase velocity of a Rayleigh wave at the boundary of a Poisson half-space.

It has the "obvious" solution $y = 1/4$, which is not acceptable, since (59) y must be greater than 1. Factoring $(1 - 4\,y)$ out of the LHS of (64*b*), one gets

$$(1 - 4\,y)\cdot\left(\frac{8}{3}\,y^2 - 4\,y + 1\right) = 0 \qquad (65)$$

whose other solutions are given by

$$8\,y^2 - 12\,y + 3 = 0\,; \qquad y = \frac{3 \pm \sqrt{3}}{4} \qquad (66)$$

of which only the top (+) sign is acceptable, since y must be greater then 1 (59). Finally

$$p\cdot\beta = \frac{\sqrt{3 + \sqrt{3}}}{2} = 1.0877 \qquad (67)$$

which is often expressed as the ratio of the phase velocity $C = 1/p$ of the wave to the shear velocity

$$\frac{C}{\beta} = \frac{2}{\sqrt{3 + \sqrt{3}}} = 0.9194 \qquad (68)$$

A classical Rayleigh wave consists of the superposition of an inhomogenous P wave and an inhomogeneous S wave, which have exactly the right combination of wave vectors to satisfy both boundary conditions without the need of a wave propagating in the opposite z direction, which would be illegitimate as it would blow up at infinite positive values of z.

This solution can exist only for ONE value of the phase velocity C (68).

NOTE that the results of this chapter require

- a flat boundary
- a plane wave

→ Unfortunately, neither of these assumptions works well in real life. The Earth is spherical and point sources generate spherical waves...

The following chapters will try to address these problems.

HOMEWORK 6

One sentence answers for each question

- Can there be Rayleigh waves at the free surface of a liquid half-space? Why?
- Can there be interface [Rayleigh-type] waves at the flat boundary between two liquid half-spaces? Why ?
- Can there be Love [*i.e.,* $SH-$type Rayleigh] waves at the free surface of a solid half-space? Why ?