E.A. OKAL *Geological Sciences 462*

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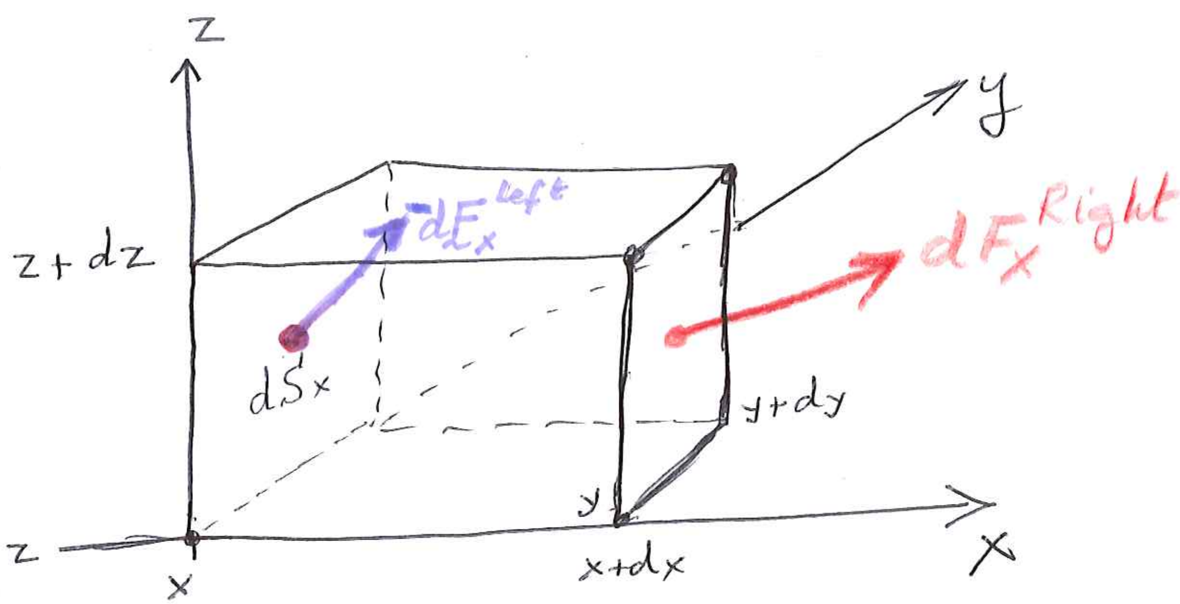
Chapter 2

Vector-wave equations and Green-function solutions for homogeneous media

Aki and Richards [1980]; *Beginning of Chapter 4.*

1. Vector wave Equation

We consider here a homogeneous, isotropic, linear elastic medium (HILE), extending to infinity. and seek to express the fundamental equation of dynamics ("$\mathbf{F} = m\,\mathbf{a}$") for an element of matter taken as a small cube (Figure 1), extending between coordinates x and $x+dx$, y and $y+dy$, z and $z+dz$:



We first focus on the force applied on the left side of the cube. According to the definition of stress, this force is

$$dF_{ix}^{Left} = -\sigma_{ix} \cdot dS_x = -\sigma_{ix} \cdot dy\,dz \tag{1}$$

The negative sign comes from the fact that the *outgoing* normal to the surface is directed towards negative x. Note also that (1) actually represents three equations, one for each component i of the force $d\mathbf{F}$.

The force applied to the right side of the cube is, similarly,

$$dF_{ix}^{Right} = \sigma_{ix}(x+dx) \cdot dy\,dz \tag{2}$$

With respect to (1), we have a positive sign, because this time the outgoing normal is in the direction of positive x, and we have to consider the value of the stress at the abscissa $(x+dx)$ of the right face of the cube.

The resultant of (1) and (2) is:

$$dF_{ix} = \left[\sigma_{ix}(x+dx) - \sigma_{ix}(x)\right] \cdot dy\, dz = \frac{\partial \sigma_{ix}}{\partial x} \cdot dx\, dy\, dz = \sigma_{ix,x} \cdot dx\, dy\, dz \qquad (3)$$

If we then add the contributions of the four other faces, we obtain

$$dF_i = \left[\sigma_{ix,x} + \sigma_{iy,y} + \sigma_{iz,z}\right] \cdot dx\, dy\, dz = \sigma_{ik,k} \cdot dx\, dy\, dz \qquad (4)$$

so that the *volume density of force* $\mathbf{f}^{Internal}$, defined by $d\mathbf{F} = \mathbf{f}^{Internal} \cdot dV = \mathbf{f}^{Internal} \cdot dx\, dy\, dz$, is just given by:

$$f_i^{Internal} = \sigma_{ik,k} \qquad \text{or} \qquad \mathbf{f}^{Internal} = \mathbf{div}\ \sigma \qquad (5)$$

Note that in (5), the operator **div** is written in bold, since the divergence of a tensor is a vector.

For any given stress field $\sigma\,(x, y, z)$, (5) represents, at each point, the density of restoring forces, seeking to bring the material back to its equilibrium state.

In a HILE medium, we apply Hooke's law,

$$\sigma_{ik} = \lambda\, \varepsilon_{ll}\, \delta_{ik} + 2\mu\, \varepsilon_{ik} \qquad (6)$$

(where λ and μ are the Lamé constants of the material), to express σ as a function of the strains ε, and hence of the spatial derivatives of the displacement field **u**, and then apply Newton's law to obtain a partial differential equation ("wave equation") involving only **u**.

In addition, we reserve the possibility of a field of *External* density of forces (*e.g.*, gravity), $\mathbf{f}^{External}$.

$$\rho\, \ddot{u}_i = \sigma_{ik,k} + f_i^{External} = f_i + \lambda\, \varepsilon_{ll,i} + 2\mu\, \varepsilon_{ik,k} = f_i + \lambda\, u_{l,il} + \mu\left[u_{i,kk} + u_{k,ik}\right] \qquad (7)$$

(where we drop the superscript *External*)

We can rewrite (7) as : †

$$\rho\, \ddot{u}_i = f_i + (\lambda+2\mu)\, u_{k,ki} - \mu\left[u_{i,kk} - u_{k,ki}\right] \qquad (8a)$$

$$\rho\, \ddot{\mathbf{u}} = \mathbf{f} + (\lambda+2\mu)\, \mathbf{grad}\, \mathrm{div}\, \mathbf{u} - \mu\, \mathbf{curl}\, \mathbf{curl}\, \mathbf{u} \qquad (8b)$$

† Why? .... The end justifies the means !

HOMEWORK 1

- Prove (8*b*) from (8*a*).

  Hint:

  In tensor notation, the curl of a vector is given by

  $$(\textbf{curl V})_i = \varepsilon_{ijk}\, u_{k,j} \qquad (8c)$$

  where the ε_{ijk} make up the permutation tensor: $\varepsilon_{ijk} = 0$ if any two indices are equal, +1 if *ijk* is a direct permutation (*e.g.*, *xyz*), −1 if it is an indirect one (*e.g.*, *xzy*).

  You may have to use to the identity

  $$\varepsilon_{ijk} \cdot \varepsilon_{lmk} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl} \qquad (8d)$$

  [[the 3-d permutation tensor ε_{ijk} is totally unrelated to the 2-d strain tensor ε_{ij}]].

We first consider the case $\mathbf{f} = \mathbf{0}$ (no external forces), and impose various symmetries to identify the *nature* of seismic waves. This problem will illustrate how a HILE medium, initially deformed, evolves in time and space as a consequence of this pertubation. Equation (8) becomes:

$$\rho\,\ddot{\mathbf{u}} \;=\; (\lambda + 2\mu)\,\mathbf{grad}\,\mathrm{div}\,\mathbf{u} \;-\; \mu\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} \qquad (9)$$

This equation (9) is not really as nice as we would hope, because the coefficients $\lambda + 2\mu$ and μ are not equal (can they be ?). For this reason, we use the following result:

> *Any vector field* $\mathbf{u}$ *can be decomposed as the sum of a gradient and a curl:*
>
> $$\mathbf{u} \;=\; \mathbf{grad}\,\phi \;+\; \mathbf{curl}\,\mathbf{A} \qquad \text{with} \qquad \mathrm{div}\,\mathbf{A} \;=\; 0 \qquad (10)$$
>
> → This is known as *Helmholtz' decomposition* [*Aki and Richards,* 1980, Box 4.2, p. 69]

We then split $\mathbf{u}$ accordingly:

$$\mathbf{u} \;=\; \mathbf{v} + \mathbf{w}\,; \qquad \mathbf{v} \;=\; \mathbf{grad}\,\phi\,; \qquad \mathbf{w} \;=\; \mathbf{curl}\,\mathbf{A}\;;\; \mathrm{div}\,\mathbf{A} \;=\; 0 \qquad (11)$$

$$\rho\left(\mathbf{grad}\,\ddot{\phi} \;+\; \mathbf{curl}\,\ddot{A}\right) \;=\; (\lambda + 2\mu)\,\mathbf{grad}\,\mathrm{div}\,\mathbf{grad}\;\phi \;+\; \mu\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{A} \qquad (12a)$$

which we rewrite as (remember "H" in HILE):

$$\mathbf{grad}\left[\,\rho\,\ddot{\phi} \;-\; (\lambda + 2\mu)\,\mathrm{div}\,\mathbf{grad}\;\phi\,\right] \;+\; \mathbf{curl}\left[\,\rho\ddot{\mathbf{A}} \;+\; \mu\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{A}\,\right] \;=\; \mathbf{X} \;=\; \mathbf{0} \qquad (12b)$$

and we apply Box (10) to the null vector $\mathbf{X}$, obtaining

$$\rho\,\ddot{\phi} \;=\; (\lambda + 2\mu)\,\mathrm{div}\,\mathbf{grad}\;\phi \;=\; (\lambda + 2\mu)\,\Delta\phi \qquad (13a)$$

$$\rho\,\ddot{\mathbf{A}} \;=\; -\,\mu\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{A} \qquad (13b)$$

It is time to remember Equation (28) of Chapter 1 for any vector $\mathbf{V}$

$$\Delta\mathbf{V} \;=\; \mathbf{grad}\,\mathrm{div}\,\mathbf{V} \;-\; \mathbf{curl}\,\mathbf{curl}\,\mathbf{V} \qquad (1\_28)$$

Because, in (10), we have imposed $\mathrm{div}\,\mathbf{A} = 0$, we see that (13*b*) can be written

$$\rho\,\ddot{\mathbf{A}} \;=\; \mu\,\Delta\mathbf{A} \qquad (13c)$$

The combination of (13*a*) *and* (13*c*) *expresses that the field of displacement* $\mathbf{u}$ *can be written as the superposition of a gradient* $\mathbf{v}$ *and a curl* $\mathbf{w}$ *whose potentials* ϕ *and* $\mathbf{A}$ *satisfy* **WAVE EQUATIONS** *(**P** WAVES and **S** WAVES), with velocities* α *and* β *given by*

$$\alpha\,\sqrt{\frac{\lambda + 2\mu}{\rho}} \;=\; \sqrt{\frac{K + \frac{4}{3}\mu}{\rho}}\;; \qquad \Delta\phi \;=\; \frac{1}{\alpha^2}\,\ddot{\phi} \qquad (14a)$$

$$\beta\,\sqrt{\frac{\mu}{\rho}}\;; \qquad \Delta\mathbf{A} \;=\; \frac{1}{\beta^2}\,\ddot{\mathbf{A}} \qquad (14b)$$

2. Solutions to the Wave Equations

A. *Plane waves*

We call plane waves solutions to (14) whose spatial dependence is of the form $\hat{\nu} \cdot \mathbf{x}$ where $\hat{\nu}$ is a constant unit vector in space.

It is clear that solutions to (14*a*) are of the form

$$\phi \;=\; \phi_1\left(t - \frac{\hat{\nu} \cdot \mathbf{x}}{\alpha}\right) + \phi_2\left(t + \frac{\hat{\nu} \cdot \mathbf{x}}{\alpha}\right) \tag{15}$$

where ϕ_1 and ϕ_2 are arbitrary functions. In principle, such arbitrary functions can always be expanded onto their *Fourier* components, so it is interesting to consider the special cases when $\phi_{1\,or\,2}(\tau) \;=\; \exp i\,\omega\,\tau$, leading to

$$\phi(x,t) \;=\; \phi_0 \cdot \exp i\,[\omega t - \mathbf{k} \cdot \mathbf{x}\,] \tag{16}$$

where $\mathbf{k}$ is the wave vector:

$$\mathbf{k} \;=\; \pm\frac{\omega}{\alpha}\,\hat{\nu} \tag{17}$$

The upper sign represent a wave propagating in the direction $\hat{\nu}$, the lower one a wave propagating in the opposite direction.

Note that the displacement $\mathbf{u}$ satisfies

$$\mathbf{u} \;=\; \mathbf{grad}\,\phi \;=\; -\,i\,\phi\,\mathbf{k} \tag{18}$$

leading to this important result

THE DISPLACEMENT IN A P WAVE IS PARALLEL TO THE PROPAGATION

For an S wave, (15), (16), (17), (18) are replaced with

$$\mathbf{A} \;=\; \mathbf{A}_1\left(t - \frac{\hat{\nu} \cdot \mathbf{x}}{\beta}\right) + \mathbf{A}_2\left(t + \frac{\hat{\nu} \cdot \mathbf{x}}{\beta}\right) \tag{19}$$

$$\mathbf{A}(x,t) \;=\; \mathbf{A}_0 \cdot \exp i\,[\omega t - \mathbf{k} \cdot \mathbf{x}\,] \tag{20}$$

$$\mathbf{k} \;=\; \pm\frac{\omega}{\beta}\,\hat{\nu} \tag{21}$$

$$\mathbf{u} \;=\; \mathbf{curl}\,\mathbf{A} \;=\; -\,i\,\mathbf{k} \times \mathbf{A} \tag{22}$$

so that now,

THE DISPLACEMENT IN A S WAVE IS PERPENDICULAR TO THE PROPAGATION

Finally, $\mathbf{u}$ being a gradient in a P wave and a curl in an S wave, the displacement of a P wave is irrotational (curl-free), and that of an S wave divergence-free (*i.e.,* there is no change in volume).

B. SPHERICAL WAVES

The situation becomes a little more complex. We consider the case of the curl-free field of displacements

$$\mathbf{u} = \mathbf{v} = \mathbf{grad}\,\phi \tag{23}$$

ϕ must satisfy the wave equation (14*a*), which in spherical coordinates, takes the form

$$\Delta\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\zeta^2} = \frac{1}{\alpha^2}\frac{\partial^2\phi}{\partial t^2} \tag{24}$$

(We have replaced the azimuthal coordinate ϕ (longitude) with ζ to distinguish it from the potential ϕ.)

A spherically symmetric wave is one with only r dependence, where all derivatives with respect to θ and ζ are zero. Then

$$\frac{1}{\alpha^2}\ddot{\phi} = \left[\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r}\right] \tag{25}$$

Defining $\Phi = r\,\phi$, we obtain

$$\frac{\partial\Phi}{\partial r} = \phi + r\frac{\partial\phi}{\partial r}\,; \qquad \frac{\partial^2\Phi}{\partial r^2} = r\frac{\partial^2\phi}{\partial r^2} + 2\frac{\partial\phi}{\partial r} \tag{26}$$

hence

$$\frac{\partial^2\Phi}{\partial r^2} = \frac{1}{\alpha^2}\frac{\partial^2\Phi}{\partial t^2} \tag{27}$$

Φ satisfies a classical propagation equation, and

$$\phi(r,t) = \frac{1}{r}\left[f_1(r-\alpha t) + f_2(r+\alpha t)\right] \tag{28}$$

HOWEVER, things get more complex for the displacement $\mathbf{u} = \mathbf{grad}\,\phi$. Assume only the outgoing potential f_1 / r and drop the index 1:

$$u_r = \frac{\partial\phi}{\partial r} = \frac{1}{r}f'(r-\alpha t) - \frac{1}{r^2}f(r-\alpha t) \tag{29}$$

THE DISPLACEMENT FIELD HAS TWO TERMS:

- A far-field term (1st one), which decays as $1/r$, and involves the derivative of f (hence high frequencies);
- A near-field term (2nd one), decaying as $1/r^2$, and involving lower frequencies.

→ The displacement remains in the direction of propagation, since there is no dependence of ϕ (potential) on θ and ζ, and hence u_θ and u_ζ are identically zero.

C. CYLINDRICAL WAVES

In cylindrical symmetry, we seek a wave which will depend only on the radial component r, but neither on elevation z nor azimuth ζ (again, we change the notation so as not to confuse with the potential ϕ).

Again, we consider a curl-free wave, and seek a potential ϕ depending only on r (and t). The Laplacian (1_27) takes the simplified form

$$\Delta \phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) \tag{30}$$

leading to

$$\frac{1}{\alpha^2} \ddot{\phi} = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \tag{31}$$

In order to proceed further, we assume a harmonic dependence with time:

$$\phi(r,t) = \phi_0(r)\, e^{i \omega t} \tag{32}$$

and obtain

$$\frac{d^2 \phi_0}{dr^2} + \frac{1}{r} \frac{d\phi_0}{dr} + \frac{\omega^2}{\alpha^2} \phi_0 = 0 \tag{33}$$

Defining

$$R = r \frac{\omega}{\alpha} \tag{34}$$

and droppping the subscript 0 in ϕ yields

$$\frac{d^2 \phi}{dR^2} + \frac{1}{R} \frac{d\phi}{dR} + \phi = 0 \tag{35}$$

which is a Bessel equation (of order zero; *Abramowitz & Stegun,* 9.1.1, p. 358), whose solutions are

$$\phi = A\, J_0(R) + B\, N_0(R) \tag{36}$$

where J_0 and N_0 (Y_0, *A&S*) are the Bessel and Neumann functions of order 0, respectively.

When r, and therefore R, $\rightarrow \infty$

$$J_0(R) = \sqrt{\frac{2}{\pi R}} \cdot \left[\cos\left(R - \frac{\pi}{4}\right) + O\left(\frac{1}{R}\right) \right]; \qquad N_0(R) = \sqrt{\frac{2}{\pi R}} \cdot \left[\sin\left(R - \frac{\pi}{4}\right) + O\left(\frac{1}{R}\right) \right] \tag{37}$$

and if one defines the Hankel functions of order 0 and 1st and 2nd types

$$H_0^1(R) = J_0(R) + i\, N_0(R)\ ; \qquad H_0^2(R) = J_0(R) - i\, N_0(R) \tag{38}$$

the general solution (36) can be rewritten as a superposition of Hankel functions which **in the far field** will behave as

$$\phi(r,t) = \frac{C}{\sqrt{R}} \cdot \exp i\, \omega \left(t - \frac{r}{\alpha} \right) + \frac{D}{\sqrt{R}} \cdot \exp i\, \omega \left(t + \frac{r}{\alpha} \right). \tag{39}$$

They look like waves propagating outwards or inwards at the velocity α, and with an amplitude decaying like $\sqrt{R}$.

However, at close distances, ϕ will incorporate other terms, and then remember that actual displacements will be given by **grad** ϕ....

B. GREEN's FUNCTION SOLUTIONS

We now want to address the fundamental problem of the waves excited by an earthquake source in an infinite HILE medium. Obviously, we have a very long way to go, in particular because the geometrical representation of an earthquake source involves a *double-couple*, which is a very complex combination of forces. In addition, this earthquake source will have a time dependence, which may be a δ function, or a sudden step (Heaviside function $H(t)$, or perhaps a slow Ramp $Ra(t)$...

We proceed by first reverting to Equation (8) and assuming that this "external" force will be characterized by a *density* of force

$$\mathbf{f} = \delta(\mathbf{x})\, \delta(t) \cdot \hat{\mathbf{e}}_1 \tag{40}$$

that is to say that we consider a point source **F** at the origin of coordinates, and directed along the axis 1 of a cartesian coordinate system. We know that this is not a proper representation of a seismic source, but we hope to consider this as a Green's function, and then build from this model [*Aki and Richards,* 1980; (4.1) p.64]. We then have to solve

$$\rho\, \ddot{\mathbf{u}} - (\lambda + 2\mu)\, \mathbf{grad}\, \mathrm{div}\, \mathbf{u} + \mu\, \mathbf{curl}\, \mathbf{curl}\, \mathbf{u} = \delta(\mathbf{x})\, \delta(t) \cdot \hat{\mathbf{e}}_1 \tag{41}$$

In order to solve this problem, we follow Aki and Richards (pp. 64 sq.), whom we will largely paraphrase here. The idea is going to be to not only decompose **u** into a gradient and a curl, but **f** as well, with the hope that (41) will then separate completely into a curl-free equation and a divergence-free one. This is known as Lamé's decomposition (and theorem), (*A&R* 4.1.1 pp. 68–69).

We start by considering the simpler **scalar** equation

$$\ddot{g} - \alpha^2\, \Delta g = \delta(\mathbf{x})\, \delta(t) \tag{42}$$

[This is pretty much as simple as the problem can get: a scalar function for an excitation which is a combination of δ functions in time and space.]

The solution to (42) with zero initial conditions is

$$g(\mathbf{x}; t) = \frac{1}{4\pi\, \alpha^2\, r} \cdot \delta\left(t - \frac{r}{\alpha} \right) \tag{43}$$

where r is the spherical polar radius to point $\mathbf{x}$: $r = \sqrt{\mathbf{x} \cdot \mathbf{x}}$.

The derivation is given in Aki and Richards' Box **4.1**, *which is reproduced on Page 9.*

Note however the notation c (more general than α).

The fundamental steps in the box are:

[*Aki and Richards*, 1980; pp. 65–66]

BOX **4.1**

Proof that

$$g(\mathbf{x}, t) = \frac{1}{4\pi c^2 |\mathbf{x}|} \delta\left(t - \frac{|\mathbf{x}|}{c}\right)$$

is the solution of $\ddot{g} = \delta(\mathbf{x})\delta(t) + c^2\nabla^2 g$ *with zero initial conditions*

By symmetry, the spatial dependence of the solution can be only on the distance $r = |\mathbf{x}|$ from the source, so we seek the functional form of $g = g(r, t)$. Expressing ∇^2 as a differential operator in spherical polar coordinates, it follows that

$$\nabla^2 g = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial g}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rg).$$

Therefore, everywhere except at $r = 0$, rg satisfies the one-dimensional wave equation $(rg)'' = r\ddot{g}/c^2$ (a prime here denoting $\partial/\partial r$), and this has the well-known general solution $rg = f(t - r/c) + h(t + r/c)$. We know that $h \equiv 0$, because the required solution is outgoing, hence it remains to prove that $f(\tau) = \delta(\tau)/(4\pi c^2)$, i.e., that $4\pi c^2 f(\tau)$ has the same properties as $\delta(\tau)$ when integrated over ranges of time.

We can establish this required result by investigating the function

$$F(r, \varepsilon_1, \varepsilon_2) \equiv 4\pi c^2 \int_{r/c-\varepsilon_1}^{r/c+\varepsilon_2} g(r, t)\, dt = \frac{4\pi c^2}{r} \int_{-\varepsilon_1}^{\varepsilon_2} f(\tau)\, d\tau.$$

Operating with ∇^2 on F, we have to differentiate the limits and the integrand g with respect to r, finding

$$\nabla^2 F = 4\pi c\left[2g'(r, t) + \frac{2}{r} g(r, t) + \frac{\dot{g}}{c}(r, t)\right]_{t=r/c-\varepsilon_1}^{t=r/c+\varepsilon_2} + 4\pi c^2 \int_{r/c-\varepsilon_1}^{r/c+\varepsilon_2} \nabla^2 g(r, t)\, dt.$$

Substituting $c^2\nabla^2 g = \ddot{g} - \delta(\mathbf{x})\delta(t)$, one can carry out the above integral of $\ddot{g}$ to give another term in $\dot{g}/c$ in the square bracket above. All these terms then cancel out, since $rg = f(t - r/c)$ implies $g' = -g/r - \dot{g}/c$, which leaves

$$\nabla^2 F = -4\pi\delta(\mathbf{x}) \int_{r/c-\varepsilon_1}^{r/c+\varepsilon_2} \delta(t)\, dt.$$

When the right-hand side is integrated over any volume V, whether the origin is in V or not, it yields the same result as the volume integral of $-4\pi\delta(\mathbf{x}) \int_{-\varepsilon_1}^{\varepsilon_2} \delta(t)\, dt$. Using the property $\nabla^2(1/r) = -4\pi\delta(\mathbf{x})$, it follows that

$$F(r, \varepsilon_1, \varepsilon_2) = \frac{1}{r} \int_{-\varepsilon_1}^{\varepsilon_2} \delta(t)\, dt.$$

(F does not involve an additional harmonic function, since such a function would either add another singularity at $r = 0$ or violate the property $F \to 0$ as $r \to \infty$.) From the second equality given in the definition of F, we can now see that $4\pi c^2 \int_{-\varepsilon_1}^{\varepsilon_2} f(\tau)\, d\tau = \int_{-\varepsilon_1}^{\varepsilon_2} \delta(t)\, dt$ for all $(\varepsilon_1, \varepsilon_2)$, and hence $f(\tau)$ is the required delta function.

• Use symmetry to justify that g should be sought in the form

$$g(r,t) \;=\; \frac{1}{r}\, f(t - r/c) \tag{44}$$

• Whatever that function f is, consider the integral

$$F(r,\varepsilon_1,\varepsilon_2) \;=\; 4\pi\, c^2 \int_{r/c-\varepsilon_1}^{r/c+\varepsilon_2} g(r,t)\, dt \;=\; \frac{4\pi\, c^2}{r} \int_{-\varepsilon_1}^{\varepsilon_2} f(\tau)\, d\tau \tag{45}$$

and compute its Laplacian (the end justifies the means...). For given ε_1 and ε_2, F is just a function of r, and we can write its derivatives as

$$F' \;=\; \frac{\mathrm{d}F}{\mathrm{d}r} \;=\; 4\pi\, c\Big[\, g(r,\frac{r}{c}+\varepsilon_2) - g(r,\frac{r}{c}-\varepsilon_1)\Big] \;+\; 4\pi\, c^2 \int_{r/c-\varepsilon_1}^{r/c+\varepsilon_2} \frac{\partial g(r,t)}{\partial r}\, dt \tag{46}$$

$$\frac{1}{4\pi\, c}\, F'' \;=\; \frac{\partial g}{\partial r}(r,r/c+\varepsilon_2) - \frac{\partial g}{\partial r}(r,r/c-\varepsilon_1) \;+\; \frac{1}{c}\left[\frac{\partial g}{\partial t}(r,r/c+\varepsilon_2) - \frac{\partial g}{\partial t}(r,r/c-\varepsilon_1)\right] + \tag{47}$$

$$+\left[\frac{\partial g}{\partial t}(r,r/c+\varepsilon_2) - \frac{\partial g}{\partial t}(r,r/c-\varepsilon_1)\right] \;+\; c \int_{r/c-\varepsilon_1}^{r/c+\varepsilon_2} \frac{\partial^2 g(r,t)}{\partial r^2}\, dt$$

According to (1_33), $\quad \Delta F \;=\; F'' + \dfrac{2}{r}\, F'. \quad$ Hence

$$\frac{1}{4\pi\, c}\,\Delta F \;=\; \left[\, 2\,\frac{\partial g}{\partial r}(r,t) \;+\; \frac{2}{r}\, g(r,t) + \frac{1}{c}\frac{\partial g}{\partial t}(r,t)\right]_{t=r/c-\varepsilon_1}^{t=r/c+\varepsilon_2} \;+\; c \int_{r/c-\varepsilon_1}^{r/c+\varepsilon_2} \Delta\, g(r,t)\cdot dt \tag{48}$$

It is time to remember that g satisfies (42):

$$\Delta\, g \;=\; \frac{1}{c^2}\Big[\, \ddot{g} - \delta(\mathbf{x})\,\delta(t)\Big] \tag{42'}$$

which leads to

$$\frac{1}{4\pi\, c}\,\Delta F \;=\; 2\left[\frac{\partial g}{\partial r}(r,t) \;+\; \frac{}{r}\, g(r,t) + \frac{1}{c}\frac{\partial g}{\partial t}(r,t)\right]_{t=r/c-\varepsilon_1}^{t=r/c+\varepsilon_2} \;-\; \frac{1}{c}\,\delta(\mathbf{x})\cdot \int_{r/c-\varepsilon_1}^{r/c+\varepsilon_2} \delta(t)\cdot dt \tag{48}$$

The bracket is identically zero, because of the nature of g in (44), and so

$$\Delta F \;=\; -\,4\pi\;\; \delta(\mathbf{x})\cdot \int_{r/c-\varepsilon_1}^{r/c+\varepsilon_2} \delta(t)\cdot dt \tag{49}$$

Remember that F is a function of r, hence of $\mathbf{x}$, and that $\delta(\mathbf{x})$ is defined by

$$\delta(\mathbf{x}) \;=\; 0 \quad \text{if} \quad \mathbf{x} \neq \mathbf{0}\,; \qquad \delta(\mathbf{x}) \;=\; \infty \quad \text{if} \quad \mathbf{x} \;=\; \mathbf{0}\,; \qquad \int_{\mathbf{R}^3} \delta(\mathbf{x})\cdot \mathrm{d}^3\mathbf{x} \;=\; 1 \tag{50}$$

If we integrate (49) over the whole space ($\mathbf{R}^3$), all contributions outside of the origin ($r \neq 0$) will vanish, and the contribution from $r = 0$ will be weighted by the integral in (49), taken for the particular value $r = 0$. Remembering the property

$$\Delta\left(\frac{1}{r}\right) \;=\; -\,4\pi\;\delta(\mathbf{x}) \tag{51}$$

we obtain

$$\Delta F(r, \varepsilon_1, \varepsilon_2) = \Delta\left(\frac{1}{r}\right)\int_{-\varepsilon_1}^{\varepsilon_2} \delta(t)\cdot dt \tag{52}$$

Hence

$$F(r, \varepsilon_1, \varepsilon_2) = \frac{1}{r}\int_{-\varepsilon_1}^{\varepsilon_2} \delta(t)\cdot dt \tag{53}$$

Comparing (44), (45), and (52) shows that

$$f(t) = \frac{1}{4\pi c^2}\,\delta(t) \tag{54}$$

or, reverting to our notation (α instead of c),

$$g(\mathbf{x}; t) = \frac{1}{4\pi\alpha^2 r}\cdot\delta\left(t - \frac{r}{\alpha}\right) \tag{43}$$

Q. E. D.

(In going from (52) to (53), follow Aki and Richards for argument against adding a harmonic function to the integral F, which would violate some obvious properties of the function g .)

→ *Aki and Richards' next three problems* (pp. 66–67)

(i) Move the source in time and space — trivial

$$\ddot{g}_1 - c^2\Delta g_1 = \delta(\mathbf{x}-\xi)\cdot\delta(t-\tau): \qquad g_1(\mathbf{x};t) = \frac{1}{4\pi c^2}\cdot\frac{\delta(t-\tau-r_\xi/c)}{r_\xi} \qquad (r_\xi = |\mathbf{x}-\xi|) \tag{55}$$

(ii) Give the source a time dependence $f(t)$

$$\ddot{g}_2 - c^2\Delta g_2 = \delta(\mathbf{x}-\xi)\cdot f(t) \tag{56a}$$

Then use (55) as a Green's function, given that $f(t) = \int_{-\infty}^{+\infty} f(\tau)\delta(t-\tau)\,d\tau$, to obtain

$$g_2 = \frac{1}{4\pi c^2}\cdot\frac{f(t-r_\xi/c)}{r_\xi} \tag{56b}$$

(iii) Expand the source spatially and temporally as a function $\frac{\Phi}{\rho}$ (why over ρ, once again in anticipation...)

$$\ddot{g}_3 - c^2\Delta g_3 = \frac{\Phi(\mathbf{x},t)}{\rho} \tag{57}$$

This time, consider the double distribution integral

$$\Phi(\mathbf{x},t) = \int_{-\infty}^{+\infty} d\tau \iiint_V \Phi(\xi,\tau)\,\delta(\mathbf{x}-\xi)\,\delta(t-\tau)\,\mathrm{d}^3\xi \tag{58}$$

and use (55) and (56*b*) as Green's functions to obtain

$$g_3(\mathbf{x}, t) = \frac{1}{4\pi \rho c^2} \iiint_V \frac{\Phi\left(\xi,\ t - \dfrac{|\mathbf{x}-\xi|}{c}\right)}{|\mathbf{x}-\xi|} \cdot d^3\xi \tag{59}$$

In particular, in the case of a static field, with no time dependence (*A&R* 4.8 p. 68),

$$\phi(\mathbf{x}) = \frac{1}{4\pi \rho c^2} \iiint_V \frac{\Phi(\xi)}{|\mathbf{x}-\xi|} \cdot d^3\xi \tag{60}$$

→ *Aki and Richards'* BOX **4.2** p. 69

This represents a clever way of effecting Helmholtz' decomposition for a vector **Z**:

$$\mathbf{Z} = \mathbf{grad}\, X + \mathbf{curl}\, \mathbf{Y}\ ; \qquad \mathrm{div}\, \mathbf{Y} = 0 \tag{61}$$

Let us seek a vector field **W** such that

$$\mathbf{Z} = \Delta \mathbf{W} = \mathbf{grad}\, \mathrm{div}\, \mathbf{W} - \mathbf{curl}\, \mathbf{curl}\, \mathbf{W} \tag{62}$$

We then see that $X = \mathrm{div}\, \mathbf{W}$ and $\mathbf{Y} = \mathbf{curl}\, \mathbf{W}$ are obvious solutions to (61).

By applying (60), for example to the cartesian coordinates of the vector **Z**, we obtain

$$\mathbf{W}(\mathbf{x}) = -\frac{1}{4\pi} \iiint_V \frac{\mathbf{Z}(\xi)}{|\mathbf{x}-\xi|} \cdot d^3\xi \tag{63}$$

In principle, the volume of integration V should be the whole space ($\mathbf{R}^3$), but in practice, it can limited to a finite domain where the source is confined (*e.g.,* the solid Earth).

→ *Lamé's Theorem*

We next proceed to proving Lamé's theorem [*Aki and Richards,* 1980; (4.1.1) pp. 68 sq.]. For this purpose, we start with the fundamental equation of dynamics

$$\rho\, \ddot{\mathbf{u}} = \mathbf{f} + (\lambda + 2\mu)\, \mathbf{grad}\, \mathrm{div}\, \mathbf{u} - \mu\, \mathbf{curl}\, \mathbf{curl}\, \mathbf{u} \tag{8b}$$

and we decompose both **u** and **f** using the Helmholtz potentials

$$\mathbf{u} = \mathbf{grad}\, \phi + \mathbf{curl}\, \psi \tag{64a}$$

$$\mathbf{f} = \mathbf{grad}\, \Phi + \mathbf{curl}\, \Psi \tag{64b}$$

$$\mathrm{div}\, \psi = \mathrm{div}\, \Psi = 0 \tag{64c}$$

and we similarly decompose the initial conditions $\mathbf{u}(\mathbf{x}; 0)$ and $\dot{\mathbf{u}}(\mathbf{x}; 0)$ as

$$\mathbf{u}(\mathbf{x}; 0) = \mathbf{grad}\, C + \mathbf{curl}\, \mathbf{D} \tag{65a}$$

$$\dot{\mathbf{u}}(\mathbf{x}; 0) = \mathbf{grad}\, A + \mathbf{curl}\, \mathbf{B} \tag{65b}$$

$$\text{div}\,\mathbf{B} = \text{div}\,\mathbf{D} = 0 \tag{65c}$$

(In most applications, $A, \mathbf{B}, C, \mathbf{D}$ will be identically zero.)

Then, Lamé's theorem states that the curl-free and divergence-free parts of the motions are decoupled in a such a way that (*A&R* (4.11)-(4.13)):

$$\ddot{\phi} - \alpha^2\,\Delta\phi = \frac{\Phi}{\rho} \tag{66 (iii)}$$

$$\ddot{\psi} - \beta^2\,\Delta\psi = \frac{\Psi}{\rho} \tag{66 (iv)}$$

The proof of Lamé's theorem is given in *Aki and Richards* [1980, p. 69] by building the (admittedly akward) functions (*A&R* (4.14), (4.15))

$$\phi\,(\mathbf{x}, t) = \frac{1}{\rho}\int_0^t (t-\tau)\Big[\,\Phi(\mathbf{x},\tau) + (\lambda+2\mu)\,\text{div}\,\mathbf{u}\,(\mathbf{x},\tau)\Big]\cdot d\tau + t\,A + C \tag{67a}$$

$$\psi\,(\mathbf{x}, t) = \frac{1}{\rho}\int_0^t (t-\tau)\Big[\,\Psi(\mathbf{x},\tau) - \mu\,\mathbf{curl}\,\mathbf{u}\,(\mathbf{x},\tau)\Big]\cdot d\tau + t\,\mathbf{B} + \mathbf{D} \tag{67b}$$

Then, *A&R* (4.12) $(\text{div}\,\psi = 0)$ is indeed trivial to prove.

A&R (4.11; our 64*a*) is a little more involved. After taking the gradient and **curl** of (67*a, b*), use (8*b*) to substitute $\ddot{\mathbf{u}}$ and integrate by parts a couple of times to verify (64*a*).

To verify (66 (iii)), take a first derivative $\dot{\phi}$. It is simply

$$\dot{\phi} = \frac{1}{\rho}\int_0^t \Big[\,\Phi(\mathbf{x},\tau) + (\lambda+2\mu)\,\text{div}\,\mathbf{u}\,(\mathbf{x},\tau)\Big]\cdot d\tau + A \tag{68}$$

(the other term cancels out), so that the second derivative $\ddot{\phi}$ becomes

$$\ddot{\phi} = \frac{1}{\rho}\Big[\,\Phi(\mathbf{x},t) + (\lambda+2\mu)\,\text{div}\,\mathbf{u}\,(\mathbf{x},t)\Big] \tag{69}$$

but div **u** is also $\Delta\phi$, according to (64*a*) **Q.E.D.**,
and similarly for (66 (iv)), noting that **curl u** is also $-\Delta\psi$ **Q.E.D.**

We are now finally armed to obtain the first major Green's function (41) *for the single force* (40).

- We can directly consider a time dependence $X_0(t)$ (*A&R* 4.16 p. 70)

$$\mathbf{f}\,(\mathbf{x}) = X_0(t)\;\delta(\mathbf{x})\cdot\hat{\mathbf{e}}_1 \tag{70}$$

A direct application of (63) to the case $\mathbf{Z} = \mathbf{f}$ yields

$$\mathbf{W} = -\frac{1}{4\pi\,r}\cdot X_0(t)\cdot\hat{\mathbf{e}}_1 \tag{71}$$

$$\Phi = \text{div}\,\mathbf{W} = -\frac{X_0(t)}{4\pi}\cdot\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right) \tag{72a}$$

$$\Psi \;=\; -\,\mathbf{curl\,W} \;=\; \frac{X_0(t)}{4\pi}\;\cdot\;\left[\,0,\;\frac{\partial}{\partial x_3}\left(\frac{1}{r}\right),\;-\frac{\partial}{\partial x_2}\left(\frac{1}{r}\right)\right] \qquad (72b)$$

What Lamé's theorem states is that we can seek the curl-free ("*P*") and divergence-free ("*S*") parts of the displacements through the equations:

$$\ddot{\phi} \;-\; \alpha^2\,\Delta\phi \;=\; -\,\frac{X_0(t)}{4\pi\,\rho}\;\cdot\;\frac{\partial}{\partial x_1}\left(\frac{1}{r}\right) \qquad (73)$$

$$\ddot{\psi} \;-\; \beta^2\,\Delta\psi \;=\; \frac{X_0(t)}{4\pi\,\rho}\;\cdot\;\left[\,0,\;\frac{\partial}{\partial x_3}\left(\frac{1}{r}\right),\;-\frac{\partial}{\partial x_2}\left(\frac{1}{r}\right)\right] \qquad (74)$$

To solve (73) (with no initial conditions), we revert to (59) and obtain

$$\phi(\mathbf{x},t) \;=\; \frac{-1}{16\pi^2\,\rho\,\alpha^2}\iiint_V \frac{X_0\left(t-\dfrac{|\mathbf{x}-\xi|}{\alpha}\right)}{|\,\mathbf{x}-\xi\,|}\cdot\frac{\partial}{\partial\xi_1}\left(\frac{1}{|\xi|}\right)\cdot \mathrm{d}^3\xi \qquad (75)$$

The volume integral is computed in a system of spherical coordinates centered on the point $\mathbf{x}$, by integrating separately on spherical shells S_τ and then radially on the variable $\tau = |\,\mathbf{x}-\xi\,|\,/\,\alpha$, leading to:

$$\phi(\mathbf{x},t) \;=\; \frac{-1}{16\pi^2\,\rho\,\alpha^2}\int_0^\infty \frac{X_0\,(t-\tau)}{\tau}\cdot\left[\iint_{S_\tau}\frac{\partial}{\partial\xi_1}\left(\frac{1}{|\xi|}\right)\cdot \mathrm{d}S_\tau\right]\cdot d\tau \qquad (76)$$

In their BOX **4.3** pp.71–72 (reproduced on Pages 14–15), *Aki and Richards* [1980] show that this equation simplifies to:

$$\phi(\mathbf{x},t) \;=\; \frac{-1}{4\pi\,\rho}\cdot\left(\frac{\partial}{\partial x_1}\,\frac{1}{|\mathbf{x}|}\right)\cdot\int_0^{\frac{|\mathbf{x}|}{\alpha}} \tau\,X_0\,(t-\tau)\cdot d\tau \qquad (77)$$

and similarly for the divergence-free terms:

$$\psi(\mathbf{x},t) \;=\; \frac{1}{4\pi\,\rho}\cdot\left[\,0,\;\frac{\partial}{\partial x_3}\,\frac{1}{|\mathbf{x}|},\;-\frac{\partial}{\partial x_2}\,\frac{1}{|\mathbf{x}|}\right]\cdot\int_0^{\frac{|\mathbf{x}|}{\beta}} \tau\,X_0\,(t-\tau)\cdot d\tau \qquad (78)$$

BOX **4.3**

Evaluation of a surface integral

We define

$$h(\mathbf{x},\tau) \equiv \iint_{|\mathbf{x}-\xi|=\alpha\tau} \frac{\partial}{\partial\xi_1}\frac{1}{|\xi|}\,dS(\xi)$$

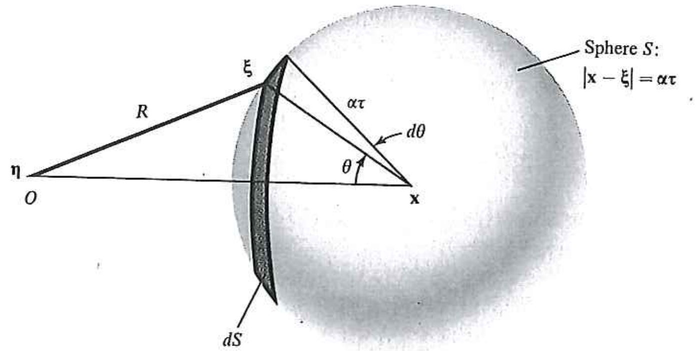
and show here that

$$h(\mathbf{x}, \tau) = 0 \qquad \text{for } \tau > |\mathbf{x}|/\alpha,$$

but

$$h(\mathbf{x}, \tau) = 4\pi\alpha^2\tau^2 \frac{\partial}{\partial x_1}\frac{1}{|\mathbf{x}|} \qquad \text{for } \tau < |\mathbf{x}|/\alpha.$$

i) Note the physical meaning of the result: suppose there is a uniform surface density on S. Then $|\xi|^{-1}\,dS$ is proportional to the gravitational potential of dS at O, and $\partial|\xi|^{-1}/\partial\xi_1 dS$ is the component of the force in the $\hat{\mathbf{x}}_1$ direction. The desired result follows from finding the total potential at O due to the shell and then differentiating to get the total force component along $\hat{\mathbf{x}}_1$. The potential inside a spherical shell is constant, and outside the spherical shell one can find the potential by lumping all the mass into a point at the center, i.e., at $\mathbf{x}$.



ii) Detailed proof: suppose that O is at $\boldsymbol{\eta}$, so that we can differentiate with respect to varying $\boldsymbol{\eta}$ and subsequently set $\boldsymbol{\eta} = \mathbf{0}$. Also take $r = |\mathbf{x} - \boldsymbol{\eta}|$, $R = |\boldsymbol{\xi} - \boldsymbol{\eta}|$, and θ as the angle between $\mathbf{x} - \boldsymbol{\eta}$ and $\mathbf{x} - \boldsymbol{\xi}$. Then

$$h = -\frac{\partial}{\partial\eta_1}\iint_S \frac{dS}{R} \qquad \text{(since } \boldsymbol{\eta} \text{ is fixed for all } \boldsymbol{\xi} \text{ on } S\text{)}.$$

Now choose $dS = 2\pi\alpha^2\tau^2 \sin\theta\, d\theta$:

$$\iint_S \frac{dS}{R} = 2\pi\alpha^2\tau^2 \int_0^\pi \frac{\sin\theta\, d\theta}{R}.$$

But $R^2 = r^2 + \alpha^2\tau^2 - 2r\alpha\tau\cos\theta$, so that $2R\,dR = 2r\alpha\tau\sin\theta\,d\theta$, and

$$\iint_S \frac{dS}{R} = \frac{2\pi\alpha\tau}{r}\int_{R(\theta=0)}^{R(\theta=\pi)} dR = \frac{2\pi\alpha\tau}{r}\int_{|\alpha\tau-r|}^{\alpha\tau+r} dR = \begin{cases} 4\pi\alpha\tau & \text{if } O \text{ is inside } S \quad (\tau > r/\alpha) \\ \dfrac{4\pi\alpha^2\tau^2}{r} & \text{if } O \text{ is outside } S \quad (\tau < r/\alpha). \end{cases}$$

Hence, if O is inside S,

$$h = -\frac{\partial}{\partial\eta_1} 4\pi\alpha\tau = 0 \qquad (\tau > r/\alpha),$$

and if O is outside S,

$$h = -\frac{\partial}{\partial\eta_1}\frac{4\pi\alpha^2\tau^2}{r} = 4\pi\alpha^2\tau^2\frac{\partial}{\partial x_1}\frac{1}{r} \qquad (\tau < r/\alpha).$$

The philosophy behind BOX **4.3** *is as follows.*

The surface integral can be thought of as representing the component along coordinate 1 of a "$1/r^2$" attractive field (*e.g.*, gravitational, electrostatic), calculated at the point $\xi = \mathbf{0}$, for a system of charges distributed on the sphere S_τ with a uniform unit density.

It is well known that the field in question is $\mathbf{0}$ if the point ξ is inside the sphere ($|\mathbf{x} - \xi| < \alpha\,\tau$), and equivalent, if the point is outside the sphere ($|\mathbf{x} - \xi| \geq \alpha\,\tau$), to that obtained by collapsing all the charge (in this case $4\pi\alpha^2\,\tau^2$) at the center of the sphere, hence calculating the field as $\dfrac{\partial}{\partial x_1}\dfrac{1}{|\mathbf{x}|}$.

Since the computation is made at $\xi = \mathbf{0}$, the value of the surface integral in (76) is immediately:

$$\iint_{S\tau} = 0 \qquad (|\mathbf{x}| < \alpha\tau) \qquad (79a)$$

$$\iint_{S\tau} = 4\pi\alpha^2\,\tau^2 \cdot \frac{\partial}{\partial x_1}\frac{1}{|\mathbf{x}|} \qquad (|\mathbf{x}| \geq \alpha\tau) \qquad (79b)$$

→ Then, all that remains to obtain the solution to our problem ("*Find the displacement field as a function of space and time when the force (70) is imposed*") is to apply (64*a*).

Note that even this is far from simple. First, set $r = \sqrt{x_i\ x_i}$, and note that

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}\ ; \qquad \frac{\partial}{\partial x_j}\frac{1}{r} = -\frac{1}{r^2}\frac{\partial r}{\partial x_j} = \frac{-x_j}{r^3} \qquad (80)$$

Then rewrite (77) and (78) by generalizing the index 1 as j (the direction of the force), obtaining

$$\phi_j(\mathbf{x},t) = \frac{1}{4\pi\,\rho}\cdot\frac{x_j}{r^3}\cdot\int_0^{\frac{r}{\alpha}} \tau\ X_0\,(t-\tau)\cdot d\tau \qquad (81)$$

$$\psi_1(\mathbf{x},t) = \frac{1}{4\pi\,\rho\,r^3}\cdot\Big[\,0\,,\ -x_3\,,\ x_2\,\Big]\cdot\int_0^{\frac{r}{\beta}} \tau\ X_0\,(t-\tau)\cdot d\tau \qquad (82a)$$

so that the coordinates of ψ_j (created by the force along j) can be written as

$$\psi_{jk}(\mathbf{x},t) = \frac{1}{4\pi\,\rho\,r^3}\cdot\varepsilon_{jlk}\,x_l\cdot\int_0^{\frac{r}{\beta}} \tau\ X_0\,(t-\tau)\cdot d\tau \qquad (82b)$$

Equation (64*a*) is then written in tensor notation (u_{ij} is the i-th component created by a force in the j-th direction:

$$u_{ij}\,(\mathbf{x},t) = \phi_{j,i} + \varepsilon_{imk}\,\psi_{jk\,,m} \qquad (83)$$

In (83), there are three contributions from (81)

$$\frac{1}{4\pi\,\rho}\cdot\frac{\delta_{ij}}{r^3}\cdot\int_0^{\frac{r}{\alpha}} \tau\ X_0\,(t-\tau)\cdot d\tau \qquad (84a)$$

$$\frac{-3}{4\pi\,\rho} \cdot \frac{x_j x_i}{r^5} \cdot \int_0^{\frac{r}{\alpha}} \tau\, X_0(t-\tau) \cdot d\tau \qquad (84b)$$

$$\frac{1}{4\pi\,\rho} \cdot \frac{x_j}{r^3} \cdot \frac{r}{\alpha} X_0(t-r/\alpha) \cdot \frac{x_i}{\alpha r} = \frac{1}{4\pi\,\rho\,\alpha^2\,r} \frac{x_i\,x_j}{r^2} X_0\left(t-\frac{r}{\alpha}\right) \qquad (84c)$$

There are, again, three groups of contributions from (82)

$$\frac{1}{4\pi\,\rho\,r^3} \cdot \varepsilon_{imk}\varepsilon_{jlk}\,x_{l,m} \cdot \int_0^{\frac{r}{\beta}} \tau\, X_0(t-\tau) \cdot d\tau \qquad (85a)$$

Remembering $\varepsilon_{imk}\varepsilon_{jlk} = \delta_{ij}\,\delta_{ml} - \delta_{il}\,\delta_{mj}$ and $x_{l,m} = \delta_{lm}$, this adds to

$$\frac{1}{4\pi\,\rho\,r^3} \cdot 2\delta_{ij} \int_0^{\frac{r}{\beta}} \tau\, X_0(t-\tau) \cdot d\tau \qquad (85a')$$

Then

$$\frac{-3}{4\pi\,\rho\,r^5} \cdot \varepsilon_{imk}\varepsilon_{jlk}\,x_l x_m \cdot \int_0^{\frac{r}{\beta}} \tau\, X_0(t-\tau) \cdot d\tau \qquad (85b)$$

Again, remembering $\varepsilon_{imk}\varepsilon_{jlk} = \delta_{ij}\,\delta_{ml} - \delta_{il}\,\delta_{mj}$, this adds up to

$$\frac{-3}{4\pi\,\rho\,r^5}\left[\delta_{ij}\,\delta_{ml}x_l x_m - x_i\,x_j\right] \cdot \int_0^{\frac{r}{\beta}} \tau\, X_0(t-\tau) \cdot d\tau \qquad (85b')$$

so that (85*a*') and (85*b*') add to

$$\frac{1}{4\pi\,\rho\,r^3}\left[\frac{3\,x_i\,x_j}{r^2} - \delta_{ij}\right] \cdot \int_0^{\frac{r}{\beta}} \tau\, X_0(t-\tau) \cdot d\tau \qquad (85a+b)$$

and finally, the third contribution from (82) is:

$$\frac{1}{4\pi\,\rho\,r^3} \cdot \varepsilon_{imk}\varepsilon_{jlk}\,x_l \cdot \frac{r}{\beta} X_0(t-r/\beta)\frac{x_m}{r\,\beta} \qquad (85c)$$

Using $\varepsilon_{imk}\varepsilon_{jlk} = \delta_{ij}\,\delta_{ml} - \delta_{il}\,\delta_{mj}$ a final time, we obtain

$$\frac{1}{4\pi\,\rho\,\beta^2\,r} \cdot \left(\delta_{ij} - \frac{x_i\,x_j}{r^2}\right) X_0\left(t-\frac{r}{\beta}\right) \qquad (85c')$$

Regrouping (84*a, b*) and (85*a*+*b*), we obtain

$$\frac{1}{4\pi\,\rho\,r^3}\left[\frac{3\,x_i\,x_j}{r^2} - \delta_{ij}\right] \cdot \int_{r/\alpha}^{r/\beta} \tau\, X_0(t-\tau) \cdot d\tau \qquad (86)$$

and changing the notation to introduce the *direction cosines*

$$\gamma_i = \frac{x_i}{r} \qquad (87)$$

we obtain the **dynamic Somigliana tensor** G_{ij} such that the i–th component of displacement at point $\mathbf{x}$ created by a force

$$\mathbf{f} = f_j\ \hat{\mathbf{e}}_\mathbf{j}\ \delta(\mathbf{x})\,X_0(t) \qquad (88)$$

will be given by

$$u_i(\mathbf{x}, t) = G_{ij}\,f_j \qquad (89)$$

Dynamic Somigliana Tensor

$$
\begin{aligned}
G_{ij} &= \frac{1}{4\pi\,\rho\,\alpha^2}\,\frac{\gamma_i\,\gamma_j}{r}\;\cdot\;X_0\left(t-\frac{r}{\alpha}\right)\;+\\
&+\frac{1}{4\pi\,\rho\,\beta^2}\,\frac{\delta_{ij}-\gamma_i\,\gamma_j}{r}\;\cdot\;X_0\left(t-\frac{r}{\beta}\right)\;+\\
&+\frac{1}{4\pi\,\rho}\cdot\frac{3\,\gamma_i\,\gamma_j-\delta_{ij}}{r^3}\cdot\int_{r/\alpha}^{r/\beta}\tau\;X_0\,(t-\tau)\cdot d\tau
\end{aligned}
\qquad (90)
$$

Fundamental properties of (90)

- The first term represents a field propagating from the source at the velocity α of P waves, and featuring the time history of the source, with an amplitude decaying as $1/r$. Note that, for a given j, u_i will be proportional to γ_i, and hence **u** is parallel to **x**.
- The second term similarly represents a field propagating from the source at the velocity β of S waves, and featuring the time history of the source, with an amplitude decaying as $1/r$. Note that the scalar product of **x** with that part of the field will involve the expression $(\gamma_i\,\delta_{ij}-\gamma_i\,\gamma_i\,\gamma_j)$ which vanishes, meaning that the displacement expressed by that term is at right angles to **x**.
- The third term is a near-field contribution, which decays faster that the previous ones. If X_0 has a finite duration in ttime, T, it is non-zero only between r/α and $(r/\beta+T)$.

HOMEWORK 2

- Derive the **static Somigliana tensor** by considering the case $X_0(t) = H(t)$ (Heaviside function), and letting $t \rightarrow \infty$.
- Show that the result can be expressed as

$$S_{ij} = \frac{1}{8\pi\mu} \cdot \left[\delta_{ij}\, r_{,ll} - \Omega\, r_{,ij} \right] \tag{91}$$

where

$$\Omega = \frac{\lambda + \mu}{\lambda + 2\mu} \tag{92}$$

NOTE that Ω *is* ***NOT*** *the Poisson ratio...*

Finally,

→ *The Double-Couple Solution*

Earthquake sources are represented by a system of forces called a double-couple, described by a moment tensor **M** given as

$$\mathbf{M} = M_0 \left[\hat{\mathbf{d}}\,\hat{\mathbf{n}}^{\mathrm{T}} + \hat{\mathbf{n}}\,\hat{\mathbf{d}}^{\mathrm{T}} \right] \tag{93}$$

In (93), $\hat{\mathbf{d}}$ is the unit vector of the direction of slip of the *hanging wall* with respect to the foot wall, and $\hat{\mathbf{n}}$ the unit vector normal to the fault plane, oriented from the foot wall to the hanging wall.

The superscript **T** indicates the transposed of a matrix.

M_0 is the scalar moment of the source (in units of dyn*cm or N*m).

Note

(i) that the distinction between hanging and foot walls is arbitrary. If they are permuted, both $\hat{\mathbf{d}}$ and $\hat{\mathbf{n}}$ change signs and **M** is unchanged.

(ii) that vectors in (93) are *right*-multiplied by transposed vectors; the result is ***NOT*** a scalar product, but rather a matrix.

(iii) that $\hat{\mathbf{d}}$ and $\hat{\mathbf{n}}$ play totally symmetric roles in (93), hence seismic waves generated by a point-source double-couple cannot be used to resolve the classical indeterminacy between focal planes ("true" and "conjugate" focal mechanisms).

In order to evaluate the displacement created by a point source moment tensor **M**, we consider a single component M_{pq} which can be interpreted as representing a single **couple** of forces F oriented in the p direction, and offset a (lever) length ΔL in the q direction, with $M_0 = F\,L$, taken in the limit $\{L \to 0;\ F \to \infty;\ M_0 \text{ const}\}$.

The excitation of the displacement field **u** by the component M_{pq} will thus be the *derivative* of the Green's function (90) with respect to the position of the source. Remember that (90) involved the *relative position* $(\mathbf{x} - \xi)$ of the field point **x** and the source ξ, so that derivatives with respect to the position of the source ξ will be the opposite of those with respect to **x**.

There follows that the field generated by **one component** $\boldsymbol{M_{pq}}$ of a moment tensor will be obtained from (90) by considering

$$-\,G_{ip,q} \tag{94}$$

which will involve a rather complex combination of terms. Some are direct derivatives of the variables r and γ_i; others involve the chain rule applied to the time dependence of the function X_0; and finally some reflect the bounds of the integral featured in the near field term...

To proceed with the computation, we first recall that

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} = \gamma_i\,; \qquad \frac{\partial \gamma_i}{\partial x_j} = \frac{\delta_{ij} - \gamma_i\,\gamma_j}{r} \tag{95}$$

- We first focus on the far-field terms (the first two lines of (90), which we rewrite substituting j with p), and consider the terms due to the derivatives of the propagation times. They will be

$$\frac{1}{4\pi\,\rho\,\alpha^3}\,\frac{\gamma_i\,\gamma_p\gamma_q}{r}\;\cdot\;\dot{X}_0\left(t-\frac{r}{\alpha}\right) \tag{96a}$$

and

$$\frac{1}{4\pi\,\rho\,\beta^3}\,\frac{(\delta_{ip}-\gamma_i\,\gamma_p)\,\gamma_q}{r}\;\cdot\;\dot{X}_0\left(t-\frac{r}{\beta}\right) \tag{96b}$$

- Next, from the same terms, we have to consider the derivatives of the terms γ_m and r. This yields

$$\frac{1}{4\pi\,\rho\,\alpha^2}\,\frac{3\,\gamma_i\,\gamma_p\,\gamma_q-\gamma_p\,\delta_{iq}-\gamma_i\,\delta_{pq}}{r^2}\;\cdot\;X_0\left(t-\frac{r}{\alpha}\right) \tag{97a}$$

and

$$\frac{1}{4\pi\,\rho\,\beta^2}\,\frac{\delta_{ip}\,\gamma_q+\delta_{iq}\,\gamma_p+\delta_{pq}\,\gamma_i-3\,\gamma_i\,\gamma_p\,\gamma_q}{r^2}\;\cdot\;X_0\left(t-\frac{r}{\beta}\right) \tag{97b}$$

- Then, we have the terms obtained by taking the derivatives of the near field. We start with the derivative of the integral, which brings:

$$\frac{1}{4\pi\,\rho\,\alpha^2}\cdot\frac{3\,\gamma_i\,\gamma_p\,\gamma_q-\delta_{ip}\,\gamma_q}{r^2}\cdot\;X_0\left(t-\frac{r}{\alpha}\right) \tag{98a}$$

and

$$-\frac{1}{4\pi\,\rho\,\beta^2}\cdot\frac{3\,\gamma_i\,\gamma_p\,\gamma_q-\delta_{ip}\,\gamma_q}{r^2}\cdot\;X_0\left(t-\frac{r}{\beta}\right) \tag{98b}$$

and follow with the derivatives of the γ_m and r terms.

$$\frac{3}{4\pi\,\rho}\cdot\frac{5\,\gamma_i\,\gamma_p\,\gamma_q-\delta_{ip}\,\gamma_q-\delta_{iq}\,\gamma_p-\delta_{pq}\,\gamma_i}{r^4}\cdot\int_{r/\alpha}^{r/\beta}\tau\;X_0\,(t-\tau)\cdot d\tau \tag{99}$$

- Regrouping all terms, we note that (97) and (98) combine with the same time dependence, sop that in the end, we obtain the final Green's function for a **moment tensor component** M_{pq} with time dependence (including scalar moment amplitude) $M_{pq}(t)$:

$$u_i\,(\mathbf{x},t) = \tag{100}$$

$$\frac{1}{4\pi\,\rho\,\alpha^3}\,\frac{\gamma_i\,\gamma_p\gamma_q}{r}\;\cdot\;\dot{M}_{pq}\left(t-\frac{r}{\alpha}\right)\;+$$

$$\frac{1}{4\pi\,\rho\,\beta^3}\,\frac{(\delta_{ip}-\gamma_i\,\gamma_p)\,\gamma_q}{r}\;\cdot\;\dot{M}_{pq}\left(t-\frac{r}{\beta}\right)\;+$$

$$\frac{1}{4\pi\,\rho\,\alpha^2}\,\frac{6\,\gamma_i\,\gamma_p\,\gamma_q-\gamma_p\,\delta_{iq}-\gamma_i\,\delta_{pq}-\gamma_q\,\delta_{ip}}{r^2}\;\cdot\;M_{pq}\left(t-\frac{r}{\alpha}\right)\;+$$

$$\frac{1}{4\pi\,\rho\,\beta^2}\,\frac{2\,\delta_{ip}\,\gamma_q+\delta_{iq}\,\gamma_p+\delta_{pq}\,\gamma_i-6\,\gamma_i\,\gamma_p\,\gamma_q}{r^2}\;\cdot\;M_{pq}\left(t-\frac{r}{\beta}\right)\;+$$

$$\frac{3}{4\pi\,\rho}\cdot\frac{5\,\gamma_i\,\gamma_p\,\gamma_q-\delta_{ip}\,\gamma_q-\delta_{iq}\,\gamma_p-\delta_{pq}\,\gamma_i}{r^4}\cdot\int_{r/\alpha}^{r/\beta}\tau\,M_{pq}\,(t-\tau)\cdot d\tau$$

Note that, in general, the actual displacement field for a seismic source would have to be summed over all components of the seismic moment tensors, *which means at least two, on account of the symmetry of (93).*

→ In these formulæ, note the presence of several kinds of terms:

- The first two terms, highlighted in **red** in (100), are prominent in the far field, since they decay in $1/r$. Furthermore, they feature the *time derivative* of the history of the moment release at the source. Given that this history is basically in the form of a step-function featuring a permanent deformation, the far-field will be in the form of an **impulse**. More generally speaking, these terms will be *high-frequency.*

  Of course, the first red term represents a P wave and the second one, an S wave.

- The next terms, highlighted in **blue** in (100), decay faster, in $1/r^2$, and feature the *same time dependence* as the moment release at the source. Thus they will prominent at closer range, and will be *lower-frequency.* In particular, they will induce a **permanent deformation.**

  Obviously, the first blue line has the properties of a P wave (causality, **u** parallel to propagation), while the second one has those of an S wave (causality, **u** perpendicular to propagation).

- The final terms, highlighted in **green** in (100), are more complex. They involve an integral of the moment release, and as such are expected to be lower frequency. In this respect, it is interesting to explore them in the case of a step function history for $M(t)$:

$$M_{pq}(t) = M_0 \cdot H(t) \tag{101}$$

where $H(t)$ is the Heaviside function ($H(t)=0$ for $t<0$; 1 for $t>0$).

Then,

$$I = \int_{r/\alpha}^{r/\beta} \tau\, M_{pq}(t-\tau) \cdot d\tau = \int_{t-r/\beta}^{t-r/\alpha} (t-\theta) \cdot H(\theta) \cdot d\theta \tag{102}$$

* If $t < r/\alpha$, *i.e.,* if the P wave has not yet arrived, all θ in the integral are negative, and thus this term is zero (not surprising in terms of causality).

* If $t > r/\beta$, *i.e.,* if [both the P and] the S wave[s] has [have] arrived, all θ values in the integral are positive, and thus

$$I = \int_{t-r/\beta}^{t-r/\alpha} (t-\theta) \cdot d\theta = \int_{r/\alpha}^{r/\beta} \tau \cdot d\tau = \frac{r^2}{2}\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right) \tag{103}$$

is a term independent of t and thus static. Note that it grows like r^2, but is divided by r^4 in (100), so that in the end, it decays like $1/r^2$.

→ *For long enough t, the only contribution of the green terms is to change the amplitude of the blue ones, so that the "near field" should be considered to be the combination of the blue and green terms.*

With this in mind, it is fundamental to understand that

> • *The displacement field created by a moment tensor has two fundamental components: a high-frequency far field mirroring the derivative of M(t) and decaying like 1/r, and a low-frequency near field mirroring M(t), but decaying like* $1/r^2$

→ **An interesting formulation of the far-fields terms of (100)**

Consider first the far-field terms of the P wave, and separate the time dependence of the moment tensor:

$$\mathbf{M}(t) = \mathbf{M} \cdot X_0(t) \; ; \qquad M_{pq}(t) = M_{pq} \cdot X_0(t) \tag{101}$$

We can rewrite the far-field P wave as

$$u_i^{P,FF}(t) = \frac{1}{4\pi\,\rho\,\alpha^3} \cdot \frac{1}{r} \cdot \dot{X}_0\left(t - \frac{r}{\alpha}\right) \cdot \left[\gamma_p\, M_{pq}\, \gamma_q\right] \cdot \gamma_i \tag{102}$$

or, defining $\hat{\gamma} = \gamma_i\, \hat{\mathbf{e}}_i$

$$\mathbf{u}^{P,FF}(t) = \frac{1}{4\pi\,\rho\,\alpha^3} \cdot \frac{1}{r} \cdot \dot{X}_0\left(t - \frac{r}{\alpha}\right) \cdot \hat{\gamma} \cdot < \hat{\gamma}^{\mathrm{T}}\, \mathbf{M}\, \hat{\gamma} > \tag{103}$$

where

$$R^P = < \hat{\gamma}^{\mathrm{T}}\, \mathbf{M}\, \hat{\gamma} > \tag{104}$$

represents a *FAR–FIELD P–WAVE RADIATION PATTERN*

varying from −1 to +1,

these values being reached only if $\hat{\gamma}$ is a principal direction of **M**.

R^P represents the scalar product of $\hat{\gamma}$ with $\mathbf{M}\,\hat{\gamma}$.

Similarly, the far-field S wave can be written as:

$$\mathbf{u}^{S,FF}(t) = \frac{1}{4\pi\,\rho\,\beta^3} \cdot \frac{1}{r} \cdot \dot{X}_0\left(t - \frac{r}{\alpha}\right) \cdot \left[\hat{\gamma} \times (\mathbf{M}\hat{\gamma} \times \hat{\gamma})\right] \tag{105}$$

Since $\mathbf{u}^{S,FF}$ has a degree of freedom in its orientation, it is not possible at this stage to define a simple R^S. Nevertheless, (105) introduces the concept.

Note that (104) and (105) cannot be simply combined into a single vector equation, because of the different terms (α^3 and β^3) in the denominator of their first RHS terms.

HOMEWORK 3

- An explosion can be described by an isotropic moment tensor

$$M_{pq} = M_0 \cdot \delta_{pq} \tag{106}$$

Show that an explosion generates no S waves in the far field, no S terms in the (blue) near field and no near field (green) terms in (100).