E.A. OKAL *Geological Sciences 462*

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Chapter 1

Review of some mathematical tools

1. Curvilinear Coordinates

- The most important thing to realize about curvilinear coordinates is that, contrary to cartesian ones, they are not directly related to *distance*; in particular since some of them represent angles, they are dimensionally different.

  If you change a curvilinear coordinate α by an amount $d\alpha$, then the relevant point in space moves by an amount (length)

$$dl = h_\alpha \cdot d\alpha \qquad (1)$$

  In the case of standard cartesian coordinates x, y, z, all h's are equal to 1. For a system of polar coordinates in 2-dimensional space, $h_r = 1$, but $h_\phi = r$. Note again the different dimension of those two coefficients.

  The h's are fundamental to all vector operations and calculus in curvilinear coordinates.

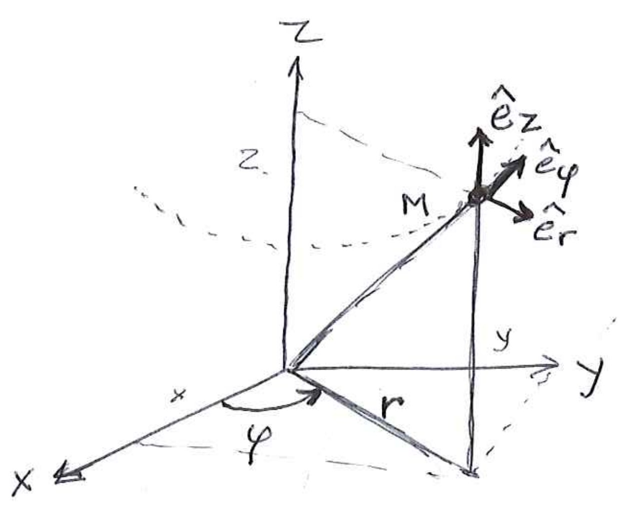
- The second important point to realize is that the unit vector frame **changes** from point to point in space, so that vector calculus becomes more delicate.

→ *Cylindrical coordinates*

In a cartesian frame (x, y, z), we keep the coordinate z, and we use a system of polar coordinates in the plane (x, y):

$$r = \sqrt{x^2 + y^2}; \qquad \phi = ATAN\_2\left[\, y, x \,\right] \qquad (2)$$

where the FORTRAN function $ATAN\_2\,(y, x)$ is the argument of the complex number $(x + i\,y)$; note that it is defined $\mathrm{mod}\, 2\pi$, as opposed to the function $\tan^{-1}(y/x)$, which is defined $\mathrm{mod}\, \pi$.



Cylindrical coordinates

Conversely,

$$x = r \cdot \cos\phi \; ; \qquad y = r \cdot \sin\phi \; ; \qquad z = z \tag{3}$$

In cylindrical coordinates, the values of the parameters h are:

$$h_r = 1 \; ; \qquad h_\phi = r \; ; \qquad h_z = 1. \tag{4}$$

→ *Spherical coordinates*

They are defined by giving the distance r of the point to the center of the system, as well as the two angles characterizing its *co-latitude* θ and *longitude* ϕ on the sphere of radius r, with respect to a [North] pole ($\theta = 0$) and a primary meridian ($\phi = 0$). Note that the geographic *latitude* λ is simply $\pi/2 - \theta$. The co-latitude θ varies from 0 to π, the longitude ϕ over a 2π interval (usually 0 to 2π, but it could be $-\pi$ to $+\pi$).

When the polar axis is oriented along the positive $\hat{\mathbf{z}}$ axis, and the primary meridian is in the plane perpendicular to the $\hat{\mathbf{y}}$ axis, spherical coordinates are given by:

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \cos^{-1} \frac{z}{r} \tag{5}$$

$$\phi = ATAN\_2 \left[\, y, x \, \right]$$

Conversely,

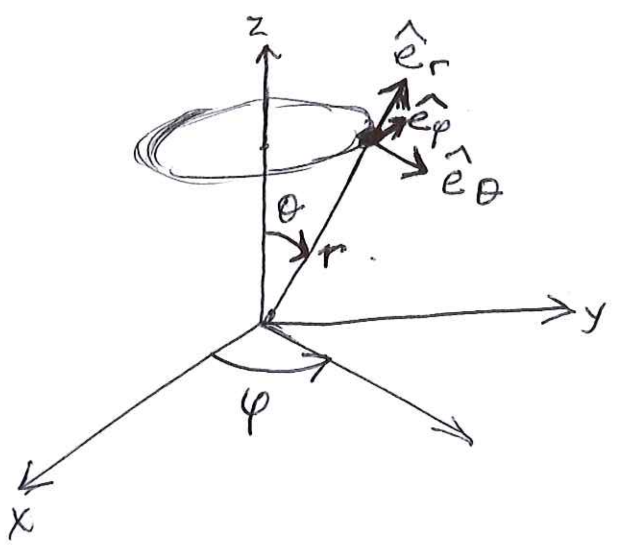
$$x = r \sin\theta \cos\phi ; \qquad y = r \sin\theta \sin\phi ; \qquad z = r \cos\theta \, . \tag{6}$$

As for the parameters h, they are given by

$$h_r = 1 \; ; \qquad h_\theta = r \; ; \qquad h_\phi = r \sin\theta \, . \tag{7}$$

The fact that h_θ and h_ϕ are not equal expresses the familiar difference in length between a degree of latitude (always equal to 111.195 km at the surface of the Earth) and a degree of longitude, which decays like $\cos\lambda$ ($\sin\theta$) when approaching the poles.

We can also define vectors $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$ and $\hat{\mathbf{e}}_\phi$ which are unit vectors in the direction corresponding to a [positive] increase in r, θ, or ϕ, respectively. Once again, this frame of vectors *depends on the particular point where it is computed.*



Spherical Coordinates

- **Transformations of vector coordinates**

We consider a field of vectors **V** (**M**) computed at a variable point **M**. In cartesian coordinates, this field is given by

$$\mathbf{V} = v_x\,\hat{\mathbf{e}}_x + v_y\,\hat{\mathbf{e}}_y + v_z\,\hat{\mathbf{e}}_z = v_i\,\hat{\mathbf{e}}_i \tag{8}$$

using tensor notation (summation implied over the dummy index i).

In curvilinear coordinates, we write similarly

$$\mathbf{V} = v_r\,\hat{\mathbf{e}}_r + v_\theta\,\hat{\mathbf{e}}_\theta + v_\phi\,\hat{\mathbf{e}}_\phi = v_\alpha\,\hat{\mathbf{e}}_\alpha \tag{9}$$

using tensor notation (summation implied over the dummy index α; Greek indices will refer to the curvilinear system, latin ones to the cartesian system).

In order to obtain the components v_α from v_i (and conversely), we equate (8) and (9), and we express the $\hat{\mathbf{e}}_i$ as a function of the $\hat{\mathbf{e}}_\alpha$ (or conversely). For example, by taking the scalar product of (8) and (9) with $\hat{\mathbf{e}}_x$, it is easy to show that

$$v_x = v_r\left[\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_x\right] + v_\theta\left[\hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{e}}_x\right] + v_\phi\left[\hat{\mathbf{e}}_\phi \cdot \hat{\mathbf{e}}_x\right] \tag{10}$$

and more generally, that

$$v_i = a_{i\alpha}\, v_\alpha \tag{11}$$

where $a_{i\alpha}$ is the scalar product between the unit vectors $\hat{\mathbf{e}}_i$ in the cartesian frame, and $\hat{\mathbf{e}}_\alpha$ in the curvilinear one. Because this is just the cosine of the angle between the two unit vectors, in other words because scalar products are permutative, one also has

$$v_\alpha = a_{i\alpha}\, v_i \tag{12}$$

with the same coefficients a (but this time the summation is over the greek index α rather than on the cartesian one i.

More generally, a second order tensor **T** (*e.g.*, a strain or stress)will transform as

$$t_{\alpha\beta} = a_{i\alpha}\, a_{j\beta}\, t_{ij}\ ; \qquad t_{ij} = a_{i\alpha}\, a_{j\beta}\, t_{\alpha\beta} \tag{13}$$

and a N-th order tensor as

$$t_{\alpha\beta\gamma\,....\,\rho} = a_{i\alpha}\, a_{j\beta}\,\cdots\cdots\, a_{s\rho}\, t_{ijk\,.....\,s} \tag{14}$$

there being the same number N of latin $(i, \cdots, s)$ and Greek $(\alpha, ..., \rho)$ indices.

Direction cosines for cylindrical coordinates

$$\begin{aligned} a_{xr} &= \cos\phi\,; & a_{x\phi} &= -\sin\phi\,; & a_{xz} &= 0\,; \\ a_{yr} &= \sin\phi\,; & a_{y\phi} &= \cos\phi\,; & a_{yz} &= 0\,; \\ a_{zr} &= 0; & a_{z\phi} &= 0; & a_{zz} &= 1. \end{aligned} \tag{15}$$

Direction cosines for spherical coordinates

$$a_{xr} = \sin\theta\cos\phi\,; \qquad a_{x\theta} = \cos\theta\cos\phi\,; \qquad a_{x\phi} = -\sin\phi\,; \tag{16a}$$

$$a_{yr} = \sin\theta \sin\phi\,; \qquad a_{y\theta} = \cos\theta \sin\phi\,; \qquad a_{y\phi} = \cos\phi\,; \tag{16b}$$

$$a_{zr} = \cos\theta; \qquad a_{z\theta} = -\sin\theta; \qquad a_{z\phi} = 0. \tag{16c}$$

- *Vector calculus*

Our goal here is to express the vector operators (**grad, curl** and div) using both the curvilinear components of the vector fields, and derivatives with respect to those coordinates. All formulæ are derived from intrinsic definitions of the vector operators

→ *THE GRADIENT*

The gradient of a function f is a vector such that upon displacement of the argument point of the fucntion from **M** to $\mathbf{M}+d\mathbf{M}$, the function varies by

$$d\,f = \mathbf{grad}\,f \cdot d\mathbf{M} \tag{17}$$

In cartesian coordinates, $d\mathbf{M} = dx_i\,\hat{\mathbf{e}}_i$, hence

$$(\mathbf{grad}\,f)_i = \frac{\partial f}{\partial x_i} = f_{,i}\ ^{\dagger} \tag{18}$$

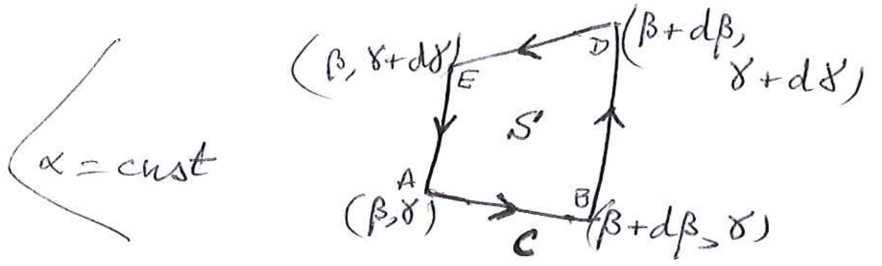
But in curvilinear coordinates ††, $d\mathbf{M} = dx_\alpha \cdot h_{\underline{\alpha}}\,\hat{\mathbf{e}}_\alpha$, so that we now have

$$(\mathbf{grad}\,f)_\alpha = \frac{1}{h_{\underline{\alpha}}}\frac{\partial f}{\partial x_\alpha} = \frac{1}{h_{\underline{\alpha}}} f_{,\alpha} \qquad \text{(no summation on } \alpha) \tag{19}$$

→ *THE CURL*

To find $(\mathbf{curl}\,\mathbf{V})_\alpha$, we apply Stokes' therorem to a little block of iso-coordinates in the β–γ plane (α = cnst):

$$I_\alpha = \oint_C \mathbf{V}\cdot d\mathbf{M} = \int\!\!\int_S (\mathbf{curl}\,\mathbf{V})_\alpha \cdot dS \tag{20}$$



Note that: $AB = h_\beta\, d\beta$
$BD = h_\gamma(\beta+d\beta, \gamma)$
$DE = -\,h_\beta(\beta, \gamma+d\gamma)$
$EA = -\,h_\gamma(\beta, \gamma)$

† In tensor notation, an index (or several indices) placed after a comma means derivation with respect to that variable.

†† In tensor notation, underlining an index means no summation over that index (or in the case of a product of three indexed terms, a simple summation over that index, of the result of the product).

It is easy to show that

$$I_\alpha = v_\beta(\beta,\gamma)\, h_\beta(\beta,\gamma)\, d\beta \; - \; V_\beta(\beta,\gamma+d\gamma)\, h_\beta(\beta,\gamma+d\gamma)\, d\beta \tag{21}$$

$$+ \; V_\gamma(\beta+d\beta,\gamma)\, h_\gamma(\beta+d\beta,\gamma)\, d\gamma \; - \; V_\gamma(\beta,\gamma)\, h_\gamma(\beta,\gamma)\, d\gamma$$

which yields;

$$(\mathbf{curl\,V})_\alpha = \frac{1}{h_\beta\, h_\gamma}\left[\frac{\partial\,(h_\gamma\, V_\gamma)}{\partial\,\beta} - \frac{\partial\,(h_\beta\, V_\beta)}{\partial\,\gamma}\right] \tag{22}$$

There are no summation conventions in (21) or (22), and the permutation $[\alpha, \beta, \gamma]$ needs to be direct.

→ *THE DIVERGENCE*

Similarly, we use Stokes' theorem to express the budget of the flux of the vector through a little cube obtained by incrementing the three curvilinear coordinates.

$$\mathrm{div}\,\mathbf{V} = \sum_\alpha \frac{1}{h_\alpha\, h_\beta\, h_\gamma}\,\frac{\partial\,(h_\beta\, h_\gamma\, v_\alpha)}{\partial\,\alpha} \tag{23}$$

In this equation, tensor notations are *not* used; The sum is over the three values of the coordinate α, and for each value, β and γ are the other two coordinates.

→ *VECTOR CALCULUS FORMULAE FOR CYLINDRICAL COORDINATES*

$$(\mathbf{grad}\,f)_r = \frac{\partial\,f}{\partial\,r}\,; \qquad (\mathbf{grad}\,f)_\phi = \frac{1}{r}\frac{\partial\,f}{\partial\,\phi}\,; \qquad (\mathbf{grad}\,f)_z = \frac{\partial\,f}{\partial\,z}\,. \tag{24}$$

$$(\mathbf{curl\,V})_r = \frac{1}{r}\frac{\partial\,v_z}{\partial\,\phi} - \frac{\partial\,v_\phi}{\partial\,z}\,; \tag{25a}$$

$$(\mathbf{curl\,V})_\phi = \frac{\partial\,v_r}{\partial\,z} - \frac{\partial\,v_z}{\partial\,r}\,; \tag{25b}$$

$$(\mathbf{curl\,V})_z = \frac{1}{r}\left[\frac{\partial\,(r\,v_\phi)}{\partial\,r} - \frac{\partial\,v_r}{\partial\,\phi}\right]. \tag{25c}$$

$$\mathrm{div}\,\mathbf{V} = \frac{1}{r}\frac{\partial\,(r\,v_r)}{\partial\,r} + \frac{1}{r}\frac{\partial\,v_\phi}{\partial\,\phi} + \frac{\partial\,v_z}{\partial\,z} \tag{26}$$

The *Laplacian* of a scalar function f is obtained from $\Delta\,f = \mathrm{div}\,\mathbf{grad}\,f$:

$$\Delta\,f = \frac{1}{r}\frac{\partial}{\partial\,r}\left(r\,\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2} \tag{27}$$

The Laplacian of a vector field $\mathbf{V}$ is defined as

$$\Delta \mathbf{V} = \mathbf{grad}\,\mathrm{div}\,\mathbf{V} - \mathbf{curl}\,\mathbf{curl}\,\mathbf{V} \tag{28}$$

and can be written as a function of the Laplacians of its components:

$$(\Delta \mathbf{V})_r = \Delta(v_r) - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi}\;; \tag{29a}$$

$$(\Delta \mathbf{V})_\phi = \Delta(v_\phi) - \frac{v_\phi}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \phi}\;; \tag{29b}$$

$$(\Delta \mathbf{V})_z = \Delta(v_z)\,. \tag{29c}$$

→ *VECTOR CALCULUS FORMULAE FOR SPHERICAL COORDINATES*

$$(\mathbf{grad}\,f)_r = \frac{\partial f}{\partial r}\;; \qquad (\mathbf{grad}\,f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}\;; \qquad (\mathbf{grad}\,f)_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\,. \tag{30}$$

$$(\mathbf{curl}\,\mathbf{V})_r = \frac{1}{r\sin\theta}\left[\frac{\partial(v_\phi \sin\theta)}{\partial \theta} - \frac{\partial v_\theta}{\partial \phi}\right]; \tag{31a}$$

$$(\mathbf{curl}\,\mathbf{V})_\theta = \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{1}{r}\frac{\partial(r\,v_\phi)}{\partial r}\;; \tag{31b}$$

$$(\mathbf{curl}\,\mathbf{V})_\phi = \frac{1}{r}\left[\frac{\partial(r\,v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta}\right]. \tag{31c}$$

$$\mathrm{div}\,\mathbf{V} = \frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \tag{32}$$

The *Laplacian* of a scalar function f is, again, obtained from $\Delta f = \mathrm{div}\,\mathbf{grad}\,f$:

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \tag{33}$$

The Laplacian of a vector field $\mathbf{V}$ is written as a function of the Laplacians of its components:

$$(\Delta \mathbf{V})_r = \Delta(v_r) - \frac{2}{r^2}\left[v_r + \frac{1}{\sin\theta}\frac{\partial(\sin\theta\, v_\theta)}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right]; \tag{34a}$$

$$(\Delta \mathbf{V})_\theta = \Delta(v_\theta) + \frac{2}{r^2}\left[\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{2\sin^2\theta} - \frac{\cos\theta}{\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}\right]; \tag{34b}$$

$$(\Delta \mathbf{V})_\phi = \Delta(v_\phi) + \frac{2}{r^2 \sin\theta}\left[\frac{\partial v_r}{\partial \phi} + \cot\theta \frac{\partial v_\theta}{\partial \phi} - \frac{v_\phi}{2\sin\theta}\right]. \tag{34c}$$

→ *APPLICATION TO STRAINS*

Strains are second order tensors, so that they will transform according to (13)

$$\varepsilon_{\alpha\beta} = a_{i\alpha}\, a_{j\beta}\, \varepsilon_{ij} \tag{35}$$

Now, remember that

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \frac{1}{2}\left(v_{ij} + v_{ji}\right) \tag{36}$$

where we define $v_{ij} = u_{i,j}$.

To compute $v_{\alpha\beta}$ as a function of the u_γ and their derivatives, simply write:

$$v_{\alpha\beta} = a_{i\alpha}\, a_{j\beta}\, u_{i,j} = a_{i\alpha}\, a_{j\beta}\, (a_{i\gamma}\, u_\gamma)_{,j} \tag{37}$$

and then invoke the chain rule to obtain

$$v_{\alpha\beta} = a_{i\alpha}\, a_{j\beta} (a_{i\gamma}\, u_\gamma)_{,\zeta} \cdot \frac{\partial x_\zeta}{\partial x_j} \tag{38}$$

$$v_{\alpha\beta} = a_{i\alpha}\, a_{j\beta}\left(a_{i\gamma}\, u_{\gamma,\zeta} + a_{i\gamma,\zeta}\, u_\gamma\right) \cdot \frac{\partial x_\zeta}{\partial x_j}$$

It is time to remember the definition of the parameters a, and in particular $a_{i\alpha}\, a_{i\gamma} = \delta_{\alpha\gamma}$, so that

$$v_{\alpha\beta} = \left(a_{j\beta}\, u_{\alpha,\zeta} + a_{i\alpha}\, a_{j\beta}\, a_{i\gamma,\zeta}\, u_\gamma\right) \cdot \frac{\partial x_\zeta}{\partial x_j} \tag{39}$$

Fortunately, most of the terms regroup or vanish; in particular, $a_{j\beta} \cdot \dfrac{\partial x_\zeta}{\partial x_j}$ is the β–component of the gradient of the function x_ζ (computed in the cartesian frame, and then rotated onto the β axis). Hence, it is just $\dfrac{1}{h_{\underline{\beta}}} \cdot x_{\zeta,\beta}$ (no sum). But the derivative $x_{\zeta,\beta}$ is obviously the Kronecker $\delta_{\zeta,\beta}$, so that

$$v_{\alpha\beta} = \frac{1}{h_{\underline{\beta}}}\left[u_{\alpha,\beta} + a_{i\alpha}\, a_{i\gamma,\beta}\, u_\gamma\right] \tag{40}$$

We give an example of the full derivation of the strain component $\varepsilon_{r\theta}$ in spherical polars, and then the full expressions (without proof) of all the strain components in cylindrical and spherical polars.

- *Compute $\varepsilon_{r\theta}$ in spherical polars*

Recall: $2\,\varepsilon_{r\theta} = v_{r\theta} + v_{\theta r}$. Then

$$v_{\theta r} = u_{\theta,r} + a_{i\theta}\, a_{i\gamma,r}\, u_\gamma \tag{41}$$

Obviously (see Eq. (16)), none of the $a_{i\alpha}$ depend on r, so the second term in (40) vanishes. As for $v_{r\theta}$, it is given by

$$v_{r\theta} = \frac{1}{r} u_{r,\theta} + a_{ir}\, a_{i\gamma,\theta}\, u_\gamma / r \tag{42}$$

Let us compute the sums (over i) $a_{ir}\, a_{i\gamma,\theta}$ for all three cases of γ:

* For $\gamma = r$, this is

$$i = x: \quad \cos\theta\cos\phi\sin\theta\cos\phi \tag{43rx}$$

$$i = y: \quad \cos\theta\sin\phi\sin\theta\sin\phi \tag{43ry}$$

$$i = z: \quad -\sin\theta\cos\theta \tag{43rz}$$

The sum vanishes.

* For $\gamma = \theta$, this is

$$i = x: \quad -\sin\theta\cos\phi\sin\theta\cos\phi \tag{43θx}$$

$$i = y: \quad -\sin\theta\sin\phi\sin\theta\sin\phi \tag{43θy}$$

$$i = z: \quad -\cos\theta\cos\theta \tag{43θz}$$

The sum equals -1.

* For $\gamma = \phi$, all three terms are zero.

In the end

$$\varepsilon_{r\theta} = \frac{1}{2}\left[u_{\theta,r} + \frac{1}{r} u_{r,\theta} - \frac{u_\theta}{r} \right] \tag{44}$$

- More generally, here are the formulæ for all strain components in cylindrical and spherical polars:

→ *CYLINDRICAL POLARS*

$$\varepsilon_{rr} = u_{r,r} \tag{45a}$$

$$\varepsilon_{\phi\phi} = \frac{1}{r} u_{\phi,\phi} + \frac{u_r}{r} \tag{45b}$$

$$\varepsilon_{zz} = u_{z,z} \tag{45c}$$

$$2\,\varepsilon_{r\phi} = u_{\phi,r} + \frac{1}{r} u_{r,\phi} - \frac{u_\phi}{r} \tag{45d}$$

$$2\,\varepsilon_{rz} = u_{r,z} + u_{z,r} \tag{45e}$$

$$2\,\varepsilon_{\phi z} \;=\; u_{\phi\,,\,z} \;+\; \frac{1}{r}\,u_{z\,,\,\phi} \tag{45f}$$

→ *SPHERICAL POLARS*

$$\varepsilon_{rr} \;=\; u_{r\,,\,r} \tag{46a}$$

$$\varepsilon_{\theta\theta} \;=\; \frac{1}{r}\,u_{\theta\,,\,\theta} \;+\; \frac{u_r}{r} \tag{46b}$$

$$\varepsilon_{\phi\phi} r \;=\; \frac{1}{r\,\sin\theta}\,u_{\phi\,,\,\phi} \;+\; \frac{u_r}{r} \;+\; \frac{\cot\theta}{r}\,u_\theta \tag{46c}$$

$$2\,\varepsilon_{r\theta} \;=\; u_{\theta\,,\,r} \;+\; \frac{1}{r}\,u_{r\,,\,\theta} \;-\; \frac{u_\theta}{r} \tag{46d}$$

$$2\,\varepsilon_{r\phi} \;=\; u_{\phi\,,\,r} \;+\; \frac{1}{r\,\sin\theta}\,u_{r\,,\,\phi} \;-\; \frac{u_\phi}{r} \tag{46e}$$

$$2\,\varepsilon_{\theta\phi} \;=\; \frac{1}{r}\,u_{\phi\,,\,\theta} \;+\; \frac{1}{r\,\sin\theta}\,u_{\theta\,,\,\phi} \;-\; \frac{\cot\theta}{r}\,u_\phi \tag{46f}$$

2. Steepest-descent and saddle-point approximation

The computation of the integral

$$J(z) = \int_C e^{z \cdot f(t)} \cdot dt \tag{47}$$

can run into significant computational problems when z is large and the product $z \cdot f$ has a large imaginary part. Small variations in t can then cause $\text{Im}(z\, f)$ to oscillate fast. Such fast oscillation means that contributions to the integral change their phase very rapidly with t, and the process becomes unstable in a numerical computation.

The steepest-descent method constitutes an attempt to compute the integral on a contour along which most of the contribution to the integral comes from a point where $\text{Re}(z\, f)$ is large and $\text{Im}(z\, f)$ stationary. (It can be shown that the two go together). Then, away from this point, $\text{Im}(z\, f)$ does oscillate, but the amplitude of $\text{Re}(z\, f)$ is small.

- **LEMMA**

  The modulus, real, and imaginary parts of an analytic function $f(z)$ cannot have absolute extrema in the complex plane.

  Proof:

  Let $f = u + i\, v$, u and v real. Suppose that u has a maximum at $z = z_0$. Then consider a small circle Γ around z_0, and compute the residue integral

  $$I = \int_\Gamma \frac{f(z)}{z - z_0} \cdot dz = \int_0^{2\pi} i\,(u + i\, v)\, d\phi = 2\, i\, \pi\, f(z_0) \tag{48}$$

  according to the residue theorem.

  If z_0 is an absolute maximum for u, it means that there exists a combination of a small strictly positive number ε and of a small number ρ such that

  $$|z - z_0| = \rho \quad \Rightarrow \quad u(z_0) - u(z) \geq \varepsilon > 0 \tag{49}$$

  If we take Γ as the circle centered on z_0 with radius ρ, then

  $$2\pi\, u(z_0) = \text{Im}(I) \leq 2\pi\,(u(z_0) - \varepsilon) < 2\pi\, u(z_0) \tag{50}$$

  the last inequality being strict, and so (49) is absurd.

  The same would occur for the imaginary part, v, of f, and for its modulus.

- *Now,* we go back to the integral (47), and we assume that there exists a point (or several points) $t = t_0$ where f is stationary (with respect to t), *i.e.*, that its derivative vanishes:

  $$f'(t) = \frac{df}{dt} = 0 \quad \text{for } t = t_0 \tag{51}$$

  Then, according to the lemma, for all values of the complex number z, the real part of the argument of the exponential in (47), $\text{Re}(z\, f(t))$, must have a saddle-point (with respect to t) at $t = t_0$, since at constant z, it is stationary, but it can have neither a maximum, nor a minimum. Around this point, and for a given z, we can write :

$$z\,f(t) \;=\; z\,f(t_0) \;+\; \frac{1}{2}\,z\,f''(t_0)\cdot(t-t_0)^2 \;+\; \ldots\ldots \tag{52}$$

Whatever the countour of integration C was in (47), we can deform it while still going through t_0.

In the vicinity of t_0, and in the complex plane, we consider different directions for the complex number $\delta t = t - t_0$. If δt is taken in the direction of the complex number $\left[z\,f''(t_0)\right]^{-1/2}$, or the opposite direction, then the real part $\mathrm{Re}\,(z\,f(t))$ increases fastest away from t_0, and the imaginary part $\mathrm{Im}\,(z\,f(t))$ is stationary. At right angles from those directions, $\mathrm{Re}\,(z\,f(t))$ decreases fastest and $\mathrm{Im}\,(z\,f(t))$ remains stationary. Along the bisectors, $\mathrm{Im}\,(z\,f(t))$ would change fastest and $\mathrm{Re}\,(z\,f(t))$ would be stationary.

We consider the path along which $\mathrm{Re}\,(z\,f(t))$ decreases fastest, and setting

$$z \;=\; |z|\cdot e^{i\phi} \tag{53}$$

we define the new variable of integration

$$\tau \;=\; \sqrt{-\,e^{i\phi}\,f''(t_0)}\cdot(t-t_0) \tag{54a}$$

$$(t-t_0)^2 \;=\; -\,\tau^2\,\frac{e^{-i\phi}}{f''(t_0)}\;; \tag{54b}$$

$$dt \;=\; \frac{d\tau}{\sqrt{-\,e^{i\phi}\,f''(t_0)}} \tag{54c}$$

Hence, the approximate value for J

$$J(z) \;=\; \frac{e^{z\,f(t_0)}}{\sqrt{-\,e^{i\phi}\,f''(t_0)}}\cdot\int_{C'} e^{-\frac{|z|}{2}\tau^2}\cdot d\tau \tag{55}$$

the contour C' being forced to feature *real* values of τ, at least in the vicinity of t_0.

If $|z| \to \infty$, the integral becomes more and more concentrated around $\tau = 0$ and takes the value $\sqrt{2\,\pi\,/\,|z|}$. Finally

$$J(z) \;=\; e^{z\,f(t_0)}\cdot\sqrt{\frac{2\,\pi}{-z\,f''(t_0)}} \;=\; e^{z\,f(t_0)}\cdot\left|\frac{2\pi}{z\,f''(t_0)}\right|^{1/2}\cdot e^{i\chi}\,, \tag{56}$$

where χ is the argument of $1/\sqrt{-z\,f''(t_0)}$, which in other words is exactly the argument of the steepest-descent path where τ is real (53).

This reproduces Formula (10) p. 205 (BOX **6.3**) of *Aki and Richards* [1980] in the case $F(\zeta) = 1$.